1. (5pts) Consider the heat equation given by the PDE:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with initial and boundary conditions:

$$u(x,0) = f(x), \qquad u(0,t) = A, \quad \text{and} \quad u(10,t) = B.$$

a. Let $u(x,t) = v(x,t) + u_e(x)$, where $u_e(x)$ solves the equilibrium solution to the PDE and the nonhomogeneous boundary conditions. Find $u_e(x)$. (Heat Equation Slides 10-12)

b. Show that $v(x,t)$ satisfies the PDE with homogeneous boundary conditions. What is the initial condition for this PDE in $v(x,t)$?

2. (6pts) The Laplacian in Cartesian coordinates is given by:

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

In polar coordinates we have:

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta).$$

Use the chain rule to show that the Laplacian in polar coordinates satisfies:

$$\nabla^2 u(x,y) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Hint: Use the chain and product rules carefully to find:

$$u_r, \quad u_{rr}, \quad \text{and} \quad u_\theta, \quad u_{\theta\theta},$$

then combine $u_{rr} + \frac{u_{\theta\theta}}{r^2}$ and using $\cos^2(\theta) + \sin^2(\theta) = 1$ and information on u_r to obtain this Laplacian in polar coordinates. (See Heat 3D Slide 11)

3. (5pts) Consider the PDE given by:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

What ordinary differential equations are implied by the method of separation of variables? (Separable Slides 6-8)