HW 11

10 10.3.1. Show that the Fourier transform is a linear operator; that is, show that

5 (a) $\mathcal{F}[c_1 f(x) + c_2 g(x)] = c_1 F(\omega) + c_2 G(\omega)$

5 (b) $\mathcal{F}[f(x)g(x)] \neq F(\omega)G(\omega)$

5 10.3.5. If $F(\omega)$ is the Fourier transform of $f(x)$, show that the inverse Fourier transform of $e^{i\omega\beta}F(\omega)$ is $f(x-\beta)$. This result is known as the **shift theorem** for Fourier transforms.

8 10.4.1. Using Green's formula, show that

$$\mathcal{F}\left[\frac{d^2 f}{dx^2}\right] = -\omega^2 F(\omega) + \frac{e^{i\omega x}}{2\pi}\left(\frac{df}{dx} - i\omega f\right)\Bigg|_{-\infty}^{\infty}.$$

20 10.4.3. *(a) Solve the diffusion equation with **convection**:

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$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} \qquad -\infty < x < \infty$$

$$u(x,0) = f(x).$$

[*Hint*: Use the convolution theorem and the shift theorem (see Exercise 10.4.5).]

10 (b) Consider the initial condition to be $\delta(x)$. Sketch the corresponding $u(x,t)$ for various values of $t > 0$. Comment on the significance of the convection term $c\,\partial u/\partial x$.

Computer Problem 3 - Fourier Integral

Consider the function

$$f(x) = \begin{cases} 0, & x < 0, \\ \pi e^{-x}, & x > 0. \end{cases}$$

The Fourier integral formula is given by

$$f(x) = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega$$

1. Find the Fourier integral coefficients $A(\omega)$ and $B(\omega)$. Give the Fourier integral representation for $f(x)$.

2. Determine what the Fourier integral converges to for all values of x.

3. The truncated Fourier integral formula is given by

   $$f(x) \approx \int_0^a [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega,$$

   where a represents the truncated wave numbers. Graph the original function and the Fourier integral representation for $a = 5$, 10, 50, and 100. Show the graph for $x \in [-10, 10]$.