1.2.9 a. (5pts) The heat balance law gives:

rate of change of heat energy in time	=	heat flowing across boundaries per unit time per unit surface area	-	heat energy flowing out of the lateral sides per unit time per surface area

$$\frac{\partial}{\partial t}\left[c\rho u(x,t)A\Delta x\right] = \Phi(x,t)A - \Phi(x+\Delta x,t)A - \omega(x,t)P\Delta x.$$

Dividing by $A\Delta x$ and assuming $c(x)$ and $\rho(x)$ constant, we have

$$c\rho\frac{\partial u}{\partial t} = \frac{\Phi(x,t)A - \Phi(x+\Delta x,t)}{\Delta x} - \frac{P}{A}\omega(x,t).$$

Taking the limit as $\Delta x \to 0$, we find

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \Phi(x,t)}{\partial x} - \frac{P}{A}\omega(x,t).$$

By Fourier's Law

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}\omega(x,t)$$

b. (2pts) With $\omega(x,t) = h(x)\left[u(x,t) - \gamma(x,t)\right]$, the equation from (a) becomes

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}\left[u(x,t) - \gamma(x,t)\right]h(x)$$

c. (2pts) The left hand side and the first part of the right hand side are the same. But instead of adding heat energy generated inside, we subtract the heat energy flowing out of the lateral surface area in (1.2.15)

d. (3pts) The circular cross-section has $P = 2\pi r$ and $A = \pi r^2$ with radius r. Constant thermal properties imply c, ρ, K_0 are constant, and $k = \frac{K_0}{c\rho}$. With $0°$ outside temperature, we have $\gamma(x,t) = 0$. now

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \frac{2\pi r}{\pi r^2}u(x,t)h(x)\frac{1}{c\rho}.$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \frac{2}{rc\rho}u(x,t)h(x).$$

e. (3pts) Assuming $h(x) = h$ and constant and that the temperature is uniform ($u(x,t) = u(t)$), the ODE is

$$u' = -\frac{2h}{rc\rho}u$$

with $u(0) = u_0$, then

$$u(t) = u_0 e^{-\frac{2h}{rc\rho}}.$$

1.4.12 a. (4pts) Fick's law (1.2.13) is $\Phi = -k\frac{\partial u}{\partial x}$. Assume a cross-section area A with initial condition $u(x,0) = f(x)$. The boundary conditions are specified flow on both ends:

$$-k\frac{\partial u}{\partial x}(0,t) = \alpha \qquad \text{and} \qquad -k\frac{\partial u}{\partial x}(L,t) = \beta.$$

The conservation law for the entire region is

$$\frac{d}{dt}\int_0^L u(x,t)dx = \Phi(0,t) - \Phi(L,t) = -k\frac{\partial u}{\partial x}(0,t) + k\frac{\partial u}{\partial x}(L,t) = \alpha - \beta.$$

In summary, the integral conservation law (change in concentration is the amount entering - amount leaving):

$$\frac{d}{dt}\int_0^L u(x,t)dx = \alpha - \beta.$$

b. (4pts) Integrating the equation for the conservation law gives $\int_0^L u(x,t)dx = (\alpha - \beta)t + c_1$. At $t = 0$, we have

$$\int_0^L u(x,0)dx = c_1 = \int_0^L f(x)dx$$

so

$$\int_0^L u(x,t)dx = (\alpha - \beta)t + \int_0^L f(x)dx.$$

Total amount of chemical in the region is

$$\int_0^L u(x,t)Adx = (\alpha - \beta)At + A\int_0^L f(x)dx.$$

c. (7pts) Assume $u(x,t) = u(x)$ in equilibrium.

$$\frac{d}{dt}\int_0^L u(x)dx = 0 = \alpha - \beta.$$

There is an equilibrium if $\alpha = \beta$. The equilibrium problem with $\alpha = \beta$ becomes

$$\frac{d^2u}{dx^2} = 0 \quad \text{with} \quad \frac{du}{dx}(0) = -\frac{\alpha}{k} \quad \text{and} \quad \frac{du}{dx}(L) = -\frac{\alpha}{k}.$$

This has the solution:

$$u(x) = c_1 x + c_2, \qquad \text{so} \qquad u'(x) = c_1 = -\frac{\alpha}{k}.$$

or

$$u(x) = -\frac{\alpha}{k}x + c_2.$$

From the initial condition,

$$\begin{aligned}
\int_0^L u(x)dx &= \int_0^L f(x)dx \\
\int_0^L \left(-\frac{\alpha}{k}x + c_2\right) dx &= \int_0^L f(x)dx \\
\left[-\frac{\alpha x^2}{2k} + c_2 x\right]_0^L &= \int_0^L f(x)dx \\
-\frac{\alpha L^2}{2k} + c_2 L &= \int_0^L f(x)dx \\
c_2 &= \frac{1}{L}\int_0^L f(x)dx + \frac{\alpha L}{2k} \\
u(x) &= -\frac{\alpha}{k}x + \frac{1}{L}\int_0^L f(x)dx + \frac{\alpha L}{2k}.
\end{aligned}$$