1. a. (2pts) We have $y_1 = e^t$, so $y_1' = e^t$ and $y_1'' = e^t$. It follows that $y_1'' - y_1 = e^t - e^t = 0$. Similarly, $y_2 = e^{-t}$ with $y_2' = -e^{-t}$ and $y_2'' = e^{-t}$. Thus, it follows that $y_2'' - y_2 = e^{-t} - e^{-t} = 0$. Thus, $y_1(t)$ and $y_2(t)$ are solutions to the differential equation.

Take a linear combination: $c_1 e^t + c_2 e^{-t} = 0$, which is equivalent to $c_1 = -c_2 e^{-2t}$. With the decaying exponential, this equality only holds for $c_1 = c_2 = 0$. Thus, the two solutions are linearly independent. (Also, can be shown using the Wronskian.)

b. (2pts) We have $y_1(t) = \sinh(t)$, so $y_1' = \cosh(t)$ and $y_1'' = \sinh(t)$. It follows that $y_1'' - y_1 = \sinh(t) - \sinh(t) = 0$. Similarly, $y_2(t) = \sinh(1-t)$, so $y_2' = -\cosh(1-t)$ and $y_2'' = \sinh(1-t)$. It follows that $y_2'' - y_2 = \sinh(1-t) - \sinh(1-t) = 0$. Thus, $y_1(t)$ and $y_2(t)$ are solutions to the differential equation.

To show that this pair forms another linearly independent set, consider $c_1\sinh(t) + c_2\sinh(1-t) = 0$. We could apply the Wronskian, but it is easier to take advantage of the fact that this must hold for all t. At $t = 0$, we have $c_2\sinh(1) = 0$, so $c_2 = 0$. At $t = 1$, we have $c_1\sinh(1) = 0$, so $c_1 = 0$. Thus, $c_1 = c_2 = 0$, and the two solutions are linearly independent.

2. a. (2pts) For the differential equation:

$$y'' - 2ay' + (a^2 + b^2)y = 0,$$

the characteristic equation satisfies:

$$\lambda^2 - 2a\lambda + (a^2 + b^2) = 0 \qquad \text{or} \qquad \lambda = a \pm ib.$$

This gives the general solution

$$y(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt) = e^{at}(c_1\cos(bt) + c_2\sin(bt)).$$

b. (3pts) The initial condition, $y(0) = y_0$, gives $y(0) = y_0 = c_1$. Differentiating (product rule), we have

$$y'(t) = e^{at}(-c_1 b\sin(bt) + c_2 b\cos(bt)) + ae^{at}(c_1\cos(bt) + c_2\sin(bt)).$$

With $y'(0) = z_0$, we obtain:

$$z_0 = c_2 b + ac_1 = bc_2 + ay_0 \qquad \text{or} \qquad c_2 = \frac{(z_0 - ay_0)}{b}.$$

It follows that the solution is:

$$y(t) = e^{at}\left(y_0\cos(bt) + \frac{(z_0 - ay_0)}{b}\sin(bt)\right).$$

c. (6pts) We consider the ODE with boundary conditions:

$$y(0) = A \qquad \text{and} \qquad y(x_0) = B.$$

From above we see that $c_1 = A$, so

$$y(t) = e^{at}(A\cos(bt) + c_2\sin(bt)).$$

The other BC implies that

$$y(x_0) = B = e^{ax_0}(A\cos(bx_0) + c_2\sin(bx_0)).$$

Provided $\sin(bx_0) \neq 0$ or equivalently, $x_0 \neq \frac{n\pi}{b}, n = 1, 2, ...$, we can uniquely solve for c_2 with

$$c_2 = \frac{B - Ae^{ax_0}\cos(bx_0)}{e^{ax_0}\sin(bx_0)},$$

which gives the **unique solution**:

$$y(t) = e^{at}\left(A\cos(bt) + \frac{(B - Ae^{ax_0}\cos(bx_0))}{e^{ax_0}\sin(bx_0)}\sin(bt)\right).$$

ii) If $\sin(bx_0) = 0$ or $x_0 = \frac{(n)\pi}{b}$, then there are infinitely many solutions provided $B - Ae^{ax_0}\cos(bx_0) = 0$, which is equivalent to

$$B = Ae^{an\pi/b}\cos(n\pi), \quad n = 1, 2, ...$$

iii) If $\sin(bx_0) = 0$ or $x_0 = \frac{(n)\pi}{b}$, then there is no solution if

$$B \neq Ae^{an\pi/b}\cos(n\pi), \quad n = 1, 2, ...$$