Spring 2023 **Math 531 Take-Home Final** **Due Thur, May 11, 2023 by Noon**

I, ________________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

For all of the problems below, perform all integrations that can be readily be done (unless told to leave them in integral form). Use orthogonality to eliminate any zero coefficients, stating which ones are zero. State clearly your reference for any short-cutted solutions to Sturm-Liouville problems.

1. Consider heat conduction in a sphere with no θ dependence given by:

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right)\right), \quad 0 < \rho < a,\ 0 \le \phi \le \pi,\ t > 0,$$

with the boundary and initial conditions:

$$u(a,\phi,t) = 0, \qquad u(\rho,\phi,0) = F(\rho)\big(2 - \cos(\phi)\big).$$

a. Use separation of variables to produce **3** ordinary differential equations. Clearly state which of these are Sturm-Liouville problems and give their appropriate boundary conditions.

b. Find the solutions to each of the ODEs. Determine the eigenvalues and eigenfunctions for the Sturm-Liouville problems. Give the orthogonality relation for each of these sets of eigenfunctions.

c. You are given that the first **4** zeroth order Legendre polynomials are:

$$P_0^0(x) = P_0(x) = 1 \qquad\qquad P_1^0(x) = P_1(x) = x$$

$$P_2^0(x) = P_2(x) = \tfrac{1}{2}(3x^2 - 1) \qquad P_3^0(x) = P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

Use the superposition principle to write the general solution, $u(\rho,\phi,t)$, as a double Fourier series. Use the information above with the initial condition to reduce the Fourier series solution. Clearly state which Fourier coefficients are **zero**.

2. Consider the nonhomogeneous partial differential equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-2t}\cos\left(\frac{2\pi x}{L}\right), \qquad t > 0 \quad \text{and} \quad 0 < x < L,$$

where $k(2\pi/L)^2 \neq 2$ and with boundary conditions:

$$\frac{\partial}{\partial x}u(0,t) = 0 \quad \text{and} \quad \frac{\partial}{\partial x}u(L,t) = 0.$$

Assume an initial condition:

$$u(x,0) = f(x).$$

Use the method of eigenfunction expansion with $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$, where $\phi_n(x)$ are the appropriate eigenfunctions corresponding to the homogeneous boundary conditions above, to solve this problem.

3. Solve the initial value problem for the nonhomogeneous heat equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) + q_0 e^{-t} r\,\sin(5\theta), \qquad 0 < r < 1, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0,$$

with the boundary conditions:

$$u(1,\theta,t) = 0, \quad u(r,0,t) = 0, \quad u_\theta(r,\pi/2,t) = 0,$$

and initial condition:

$$u(r,\theta,0) = f_0 r^3 \sin(3\theta).$$

(Note: f_0 and q_0 are constants.)

4. Consider the nonhomogeneous wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad t > 0 \quad \text{and} \quad 0 < x < \pi,$$

with one end fixed and the other having a time dependent (sinusoidal) forcing condition:

$$u(0,t) = 0 \qquad \text{and} \qquad u(\pi,t) = A\sin(\omega t).$$

Assume homogeneous initial conditions:

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

a. Find a *linear (in x) reference distribution*, $r(x,t)$, with $u(x,t) = v(x,t) + r(x,t)$, such that the PDE in $v(x,t)$ has homogeneous boundary conditions. Be sure to note the changes in both the PDE and the initial conditions with this reference function.

b. The PDE in $v(x,t)$ is nonhomogeneous, but has homogeneous boundary conditions, so apply the *method of eigenfunction expansion* with $v(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$, where $\phi_n(x)$ are the appropriate eigenfunctions corresponding to the homogeneous boundary conditions to solve this problem. The PDE in $v(x,t)$ has the form:

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} + Q(x,t), \qquad t > 0 \quad \text{and} \quad 0 < x < \pi.$$

Write the nonhomogeneous function $Q(x,t)$ in an eigenfunction expansion, $Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\phi_n(x)$, and determine the Fourier coefficients, $q_n(t)$, for this function. The problem for $v(x,t)$ will have a second order nonhomogeneous ODE in $a_n(t)$, which may have a messy expression, so can be left in integral form. (Variation of parameters solution) The expressions for the Fourier coefficients from the initial conditions can also be left in integral form, but you do need to write these integrals.

c. For certain values of c, there are unbounded solutions for $v(x,t)$. Find these values of c and explain why the solution becomes unbounded. (Resonance in the system)

5. a. Consider an infinite rod, which satisfies the partial differential equation given by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma(u - T_e), \qquad -\infty < x < \infty, \quad t > 0$$

with initial conditions:

$$u(x,0) = \begin{cases} |x|, & |x| < T_e, \\ T_e, & |x| > T_e \end{cases}.$$

Briefly describe what is happening physically to this infinite rod. Clearly explain each term and to what the initial conditions correspond.

b. Solve this problem for $u(x,t)$, showing your work.

c. Use your solution to create a 3D plot of $u(x,t)$ with $t \in [0.001, 20]$ and $x \in [-30, 30]$ with the parameters $k = 1$, $T_e = 20$, and $\gamma = 0.1$. Your program should use at least 50 terms in any Fourier series and integrate at least $\omega \in [0, 50]$ for Fourier transforms. Be sure to include your program.

6. a. Find the solution for Laplace's equation in a semi-infinite strip;

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x > 0, \quad 0 < y < b,$$

with the boundary conditions:

$$\frac{\partial}{\partial x}u(0,y) = T_0\left(\frac{b}{2} - y\right), \quad \frac{\partial}{\partial y}u(x,0) = 0, \quad u(x,b) = \begin{cases} T_0, & 0 < x < a \\ 0, & x > a \end{cases}.$$

b. Use your solution to create a 3D plot of $u(x,y)$ with $x \in [0, 20]$ and $y \in [0, 5]$ with the parameters $a = 10$, $b = 5$, and $T_0 = 25$. Your program should use at least 50 terms in any Fourier series and integrate at least $\omega \in [0, 50]$ for Fourier transforms. Be sure to include your program.