Math 531 Exam 1 Name ______________________

1. Consider the heat equation in an insulated one-dimensional rod given by:

$$\frac{\partial u}{\partial t} = 0.5\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 3, \quad t > 0,$$

with the boundary conditions and initial condition:

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(3,t) = 0, \quad u(x,0) = 6 - 4\cos(\pi x).$$

Solve this initial-boundary value problem. Find the eigenvalues and eigenfunctions for the associated Sturm-Liouville problem. What is the temperature distribution in the rod as $t \to \infty$?

Let $u(x,t) = \phi(x)h(t)$, then $\phi h' = 0.5h\phi''$ or:

$$\frac{h'}{0.5h} = \frac{\phi''}{\phi} = -\lambda.$$

The time equation is $h' = -0.5\lambda h$, so

$$h = c \cdot e^{-0.5\lambda t}.$$

The S-L problem is

$$\phi'' + \lambda\phi = 0, \qquad \phi'(0) = 0, \quad \phi'(3) = 0.$$

This is a Neumann problem, which has been shown before. The results are:
(i) If $\lambda < 0$, then only the trivial solution exists.
(ii) $\lambda_0 = 0$ is an eigenvalue with eigenfunction $\phi_0(x) = 1$.
(iii) If $\lambda > 0$, then take $\lambda = \alpha^2$. It follows that $\phi(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x)$. With $\phi'(0) = 0$, then $c_2 = 0$. From $\phi'(3) = 0$, we have $\alpha = \frac{n\pi}{3}$. It follows that the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{9}$ with eigenfunctions $\phi_n(x) = \cos\left(\frac{n\pi x}{3}\right), \quad n = 1, 2, ...$
By the superposition principle,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-0.5\left(\frac{n^2\pi^2}{9}\right)t}\cos\left(\frac{n\pi x}{3}\right).$$

The initial condition gives:

$$u(x,0) = 6 - 4\cos(\pi x) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{3}\right).$$

By orthogonality of the eigenfunctions, we obtain:

$$A_0 = 6, \qquad A_3 = -4, \qquad A_n = 0 \quad \text{for } n \neq 0, 3.$$

Thus,

$$u(x,t) = 6 - 4e^{-0.5\pi^2 t}\cos(\pi x).$$

As $t \to \infty$,

$$\lim_{t\to\infty} u(x,t) = A_0 = 6.$$

2. a. Find the eigenvalues and eigenfunctions for the Sturm-Liouville problem:

$$u'' + \lambda u = 0, \quad u(0) = 0, \quad u'(4) = 0.$$

b. Use the eigenfunctions from above to represent the function

$$f(x) = \begin{cases} 3, & 0 \le x < 2, \\ 0, & 2 \le x \le 4. \end{cases}$$

and find the Fourier coefficients.

c. To what value does the Fourier series converge at $x = 1$? At $x = 2$? At $x = -\frac{3}{2}$?

a. Have shown before in class and HW that $\lambda \le 0$ only leads to the trivial solution for this eigenvalue problem, so not an eigenvalue. (Can solve BVP directly or use Raleigh Quotient.)

So let $\lambda = \alpha^2 > 0$, then $u(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. From the BC, $u(0) = 0$, we have $c_1 = 0$. The other BC gives:

$$u'(4) = c_2 \alpha \cos(4\alpha) = 0, \qquad \text{so} \quad \alpha_n = \frac{(n - \frac{1}{2})\pi}{4}, \quad n = 1, 2, ...$$

It follows that we have eigenvectors and corresponding eigenfunctions:

$$\lambda_n = \frac{(n - \frac{1}{2})^2 \pi^2}{16} \qquad \text{and} \qquad u_n(x) = \sin\left(\frac{(n - \frac{1}{2})\pi x}{4}\right), \quad n = 1, 2, ...$$

b. The Fourier series is given by:

$$f(x) \sim \sum_{n=1}^{\infty} A_n \sin\left(\frac{(n - \frac{1}{2})\pi x}{4}\right).$$

The Fourier coefficients are given by:

$$\begin{aligned} A_n &= \frac{2}{4} \int_0^4 f(x) \sin\left(\frac{(n - \frac{1}{2})\pi x}{4}\right) = \frac{3}{2} \int_0^2 \sin\left(\frac{(n - \frac{1}{2})\pi x}{4}\right) \\ &= -\frac{12}{(2n-1)\pi} \cos\left(\frac{(n - \frac{1}{2})\pi x}{4}\right)\Bigg|_0^2 = \frac{12}{(2n-1)\pi}\left(1 - \cos\left(\frac{(n - \frac{1}{2})\pi x}{4}\right)\right) \end{aligned}$$

c. At $x = 1$, the Fourier series converges to 3 (a point of continuity).

At $x = 2$, the Fourier series converges to $\frac{3}{2}$ (midpoint between 3 and 0, the jump discontinuity).

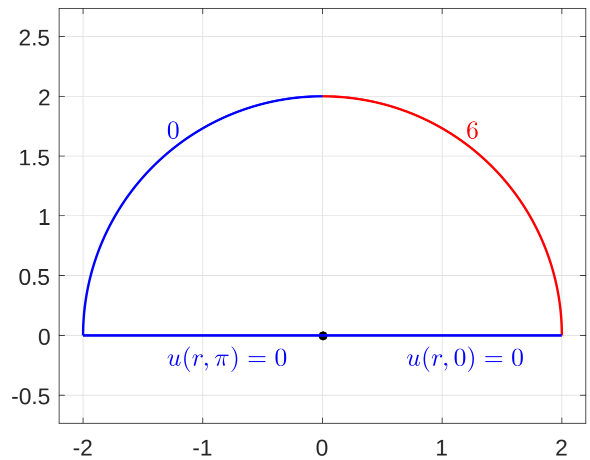
At $x = -\frac{3}{2}$, the Fourier series converges to -3. (Fourier series is the odd periodic extension).

3. Find the steady-state temperature distribution for the Figure below (assuming the faces are insulated). The region is a semi-circular region satisfying Laplace's equation:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right) = 0,$$

where the edge along the x-axis is fixed at 0. Along the semi-circular edge, we have:

$$u(2,\theta) = g(\theta) = \begin{cases} 6, & 0 < \theta < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < \theta < \pi. \end{cases}$$



Soln: Let $u(r,\theta) = h(r)\phi(\theta)$, then

$$\frac{\phi}{r}\frac{d}{dr}(rh') + \frac{h}{r^2}\phi'' = 0,$$

$$\frac{r}{h}(rh')' = -\frac{\phi'}{\phi} = \lambda.$$

The SL problem is

$$\phi'' + \lambda\phi = 0, \quad \text{with BCs} \quad \phi(0) = 0, \ \phi(\pi) = 0.$$

This is the Dirichlet problem worked in class many times. The eigenvalues are positive. The eigenvalues and corresponding eigenfunctions are given by:

$$\lambda_n = n^2 \qquad \text{and} \qquad \phi_n(\theta) = \sin(n\theta).$$

The r-equation is expanded into the Cauchy-Euler ODE with solutions $h(r) = r^\mu$:

$$r^2h'' + rh' - n^2h = 0,$$

which has the auxiliary equation $\mu(\mu-1) + \mu - n^2 = \mu^2 - n^2 = 0$, so $\mu = \pm n$. It follows that:

$$h_n(r) = c_1r^n + c_2r^{-n}.$$

The BC at the origin of solutions being bounded implies that $c_2 = 0$.

The superposition principle gives:

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^n \sin(n\theta).$$

The other boundary condition and orthogonality give

$$u(2,\theta) = \sum_{n=1}^{\infty} b_n 2^n \sin(n\theta) = g(\theta), \quad \text{so} \quad b_n 2^n \int_0^{\pi} \sin^2(n\theta)d\theta = \int_0^{\pi} g(\theta)\sin(n\theta)d\theta = 6\int_0^{\frac{\pi}{2}} \sin(n\theta)d\theta.$$

It follows that

$$b_n = \frac{12}{2^n\pi}\int_0^{\frac{\pi}{2}} \sin(n\theta)d\theta = \frac{12}{n\pi 2^n}\left(-\cos(n\theta)\Big|_0^{\frac{\pi}{2}}\right) = \frac{12}{n\pi 2^n}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right).$$

4. Consider the eigenvalue problem given by:

$$\phi'' - 2\phi' + (1+\lambda)\phi = 0, \tag{1}$$

with boundary conditions $\phi(0) = 0$ and $\phi(2) = 0$.

a. This problem becomes a Sturm-Liouville problem if it has the form:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0.$$

Make Eqn. (1) into a Sturm-Liouville problem, giving the appropriate functions $p(x)$, $q(x)$, and $\sigma(x)$ for this transformation.

b. Find the eigenvalues and eigenfunctions for this Sturm-Liouville problem. Be sure to check the different cases when $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ for eigenfunctions.

c. Let a smooth piecewise continuous function $f(x)$ be represented by a Fourier series:

$$f(x) \sim \sum_{n=1}^{\infty} b_n\phi_n(x).$$

Find an expression for b_n using the appropriate orthogonality relationship from the Sturm-Liouville problem.

a. Expand the SL problem, multiply (1) by $H(x)$, and compare terms.

$$\left.\begin{array}{r} p\phi'' + p'\phi + q\phi + \lambda\sigma\phi = 0, \\ H\phi'' - 2H\phi' + H\phi + \lambda H\phi = 0, \end{array}\right\} \qquad \begin{array}{c} p(x) = H(x) = q(x) = \sigma(x) \\ p' = H' = -2H, \quad \text{so} \quad H(x) = e^{-2x} \end{array}$$

$$\therefore \frac{d}{dx}\left(e^{-2x}\frac{d\phi}{dx}\right) + (1-\lambda)e^{-2x}\phi = 0 \qquad p(x) = q(x) = \sigma(x) = e^{-2x}.$$

b. The SL Problem is written: $e^{-2x}\left(\phi'' - 2\phi' + (1-\lambda)\phi\right) = 0$ with BCs $\phi(0) = 0 = \phi(2)$, which has the characteristic equation, $r^2 - 2r + 1 + \lambda = 0$. This has roots: $r = 1 \pm \sqrt{\lambda}$. There are 3 cases:

$$\begin{array}{l} \text{Case(i)} : \lambda = -\alpha^2 < 0, \quad \text{so} \quad \phi(x) = e^x\left(c_1\cosh(\alpha x) + c_2\sinh(\alpha x)\right), \quad \text{with} \\ \quad \phi(0) = c_1 = 0 \quad \text{and} \quad \phi(2) = e^2 c_2\sinh(2\alpha) = 0, \quad \text{so} \quad c_2 = 0, \quad \textit{i.e.}, \text{trivial solution.} \\ \text{Case(ii)} : \lambda = 0, \quad \text{so} \quad \phi(x) = (c_1 + c_2 x)e^x, \quad \text{with} \quad \phi(0) = c_1 = 0, \\ \quad \phi(2) = 2c_2e^2 = 0, \quad \text{so} \quad c_2 = 0, \quad \textit{i.e.}, \text{trivial solution.} \\ \text{Case(iii)} : \lambda = \alpha^2 > 0, \quad \text{so} \quad \phi(x) = e^x\left(c_1\cos(\alpha x) + c_2\sin(\alpha x)\right), \quad \text{with} \quad \phi(0) = c_1 = 0, \\ \quad \phi(2) = e^2 c_2\sin(2\alpha), \quad \text{giving} \quad \alpha_n = \frac{n\pi}{2}, \text{ e.v. } \lambda_n = \frac{n^2\pi^2}{4} \quad \text{and} \quad \text{e.f. } \phi_n(x) = e^x\sin\left(\frac{n\pi x}{2}\right). \end{array}$$

c. The Fourier representation is:

$$f(x) \sim \sum_{n=1}^{\infty} b_n e^x \sin\left(\frac{n\pi x}{2}\right).$$

With orthogonality, we have:

$$b_n = \frac{\int_0^2 f(x)e^x\sin\left(\frac{n\pi x}{2}\right)e^{-2x}dx}{\int_0^2\left(e^x\sin\left(\frac{n\pi x}{2}\right)\right)^2 e^{-2x}dx} = \int_0^2 f(x)e^{-x}\sin\left(\frac{n\pi x}{2}\right)dx.$$

5. a. Consider the Sturm-Liouville problem:

$$\frac{d}{d\rho}\left(\rho^2\frac{du}{d\rho}\right)+\lambda\rho^2 u=0, \qquad 1<\rho<4,$$
$$u(1)=0, \qquad u(4)=0.$$

You are given that when $\lambda=-\alpha^2<0$, two linearly independent solutions are

$$u_1(\rho)=\frac{\sinh(\alpha(\rho-1))}{\rho} \quad \text{and} \quad u_2(\rho)=\frac{\cosh(\alpha(\rho-1))}{\rho},$$

and when $\lambda=\gamma^2>0$, two linearly independent solutions are

$$u_1(\rho)=\frac{\sin(\gamma(\rho-1))}{\rho} \quad \text{and} \quad u_2(\rho)=\frac{\cos(\gamma(\rho-1))}{\rho}.$$

(You are not to show this, but must solve for $\lambda=0$.) Find the eigenvalues and eigenfunctions, and state the orthogonality relationship.

b. Let $\phi_n(\rho)$ be the eigenfunctions in Part a. Find the generalized Fourier coefficients b_n for

$$f(\rho)=\frac{5}{\rho}=\sum_{n=1}^{\infty}b_n\phi_n(\rho).$$

a. The SL problem has no complex eigenvalues, so examine the 3 real cases:
Case(i) : $\lambda=-\alpha^2<0$, so $u(\rho)=c_1\frac{\sinh(\alpha(\rho-1))}{\rho}+c_2\frac{\cosh(\alpha(\rho-1))}{\rho}$, with $u(1)=c_2=0$,
$u(4)=\frac{c_1}{4}\sinh(3\alpha)=0$, so $c_1=0$, *i.e.*, trivial solution.
Case(ii) : $\lambda=0$, so integrating the ODE gives $\rho^2\frac{du}{d\rho}=c_1$. Integrating again,
$u(\rho)=-\frac{c_1}{\rho}+c_2$. BCs imply, $u(1)=-c_1+c_2=0$ and $u(4)=-\frac{c_1}{4}+c_2=0$,
so $c_1=c_2=0$, *i.e.*, trivial solution.
Case(iii) : $\lambda=\alpha^2>0$, so $u(\rho)=c_1\frac{\sin(\alpha(\rho-1))}{\rho}+c_2\frac{\cos(\alpha(\rho-1))}{\rho}$, with $u(1)=c_2=0$,
$u(4)=\frac{c_1}{4}\sin(3\alpha)=0$, giving $\alpha_n=\frac{n\pi}{3}$, $n=1,2,....$
Thus, we have eigenvalues $\lambda_n=\frac{n^2\pi^2}{9}$ with eigenfunctions:

$$u_n(\rho)=\frac{\sin\left(\frac{n\pi(\rho-1)}{3}\right)}{\rho}, \; n=1,2,...$$

The orthogonality relationship satisfies:

$$\langle u_n,u_m\rangle=\int_1^4\frac{\sin\left(\frac{n\pi(\rho-1)}{3}\right)}{\rho}\frac{\sin\left(\frac{m\pi(\rho-1)}{3}\right)}{\rho}\cdot\rho^2\,d\rho=\begin{cases}0, & m\neq n\\ \frac{3}{2}, & m=n\end{cases}$$

b. With the eigenfunctions above, we find the generalized Fourier coefficients:

$$f(\rho) = \frac{5}{\rho} \sim \sum_{n=1}^{\infty} b_n \frac{\sin\left(\frac{n\pi(\rho-1)}{3}\right)}{\rho}.$$

We multiply both sides above by $\frac{\sin\left(\frac{m\pi(\rho-1)}{3}\right)}{\rho}\rho^2$, including the weighting factor ρ^2, then integrate from 1 to 4. It follows that:

$$\begin{aligned}
\int_1^4 \frac{5}{\rho} \frac{\sin\left(\frac{m\pi(\rho-1)}{3}\right)}{\rho} \rho^2 \, d\rho &= \int_1^4 \sum_{n=1}^{\infty} b_n \frac{\sin\left(\frac{n\pi(\rho-1)}{3}\right)}{\rho} \frac{\sin\left(\frac{m\pi(\rho-1)}{3}\right)}{\rho} \rho^2 \, d\rho \\
&= \int_1^4 b_m \frac{\sin^2\left(\frac{m\pi(\rho-1)}{3}\right)}{\rho^2} \rho^2 \, d\rho = \frac{3b_m}{2}.
\end{aligned}$$

from the orthogonality relationship, so:

$$b_m = \frac{2}{3} \int_1^4 5 \sin\left(\frac{m\pi(\rho-1)}{3}\right) d\rho = -\frac{10}{3} \frac{3}{m\pi} \left(\cos\left(\frac{m\pi(\rho-1)}{3}\right)\right)\Bigg|_1^4 = \frac{10}{m\pi}(1-(-1)^m).$$