Consider the following Sturm-Liouville problem:

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) + \lambda x\phi = 0, \qquad \phi(4) = 0.$$

There is an implicit BC that $\phi(0)$ is bounded. We want to use the eigenfunctions from this problem to find the Fourier-Bessel series to the function:

$$f(x) = \begin{cases} 0, & 0 < x \leq 1, \\ 4, & 1 < x \leq 3, \\ 0, & 3 < x \leq 4. \end{cases}$$

The ODE is Bessel's equation of order zero, so the general solution is given by

$$\phi(x) = c_1 J_0(\sqrt{\lambda}x) + c_2 Y_0(\sqrt{\lambda}x).$$

The boundedness at $x = 0$ implies $c_2 = 0$. The other BC gives $J_0(4\sqrt{\lambda}) = 0$, so the eigenvalues satisfy:

$$\lambda_n = \left(\frac{z_n}{4}\right)^2,$$

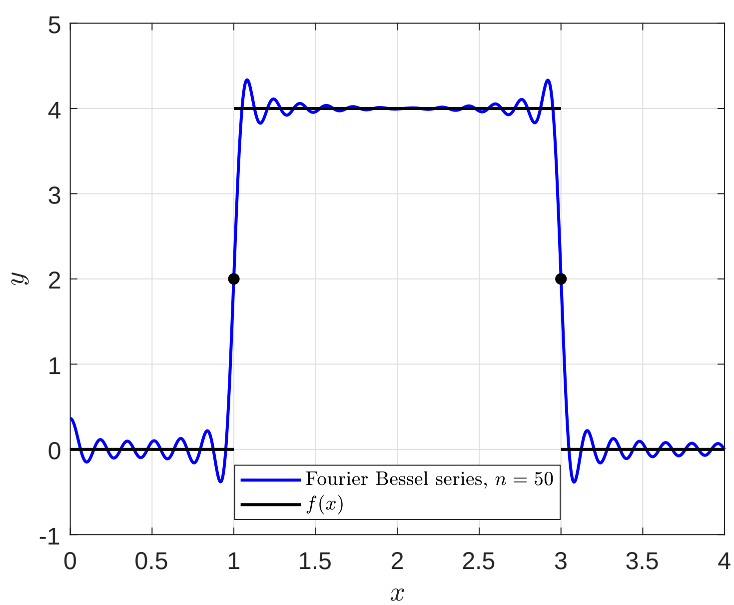
where z_n is the n^{th} zero solving $J_0(z_n) = 0$.

The Fourier-Bessel series of the function satisfies:

$$f(x) \sim \sum_{n=1}^{\infty} b_n J_0(\sqrt{\lambda_n}x).$$

Using orthogonality, the Fourier coefficients satisfy:

$$b_n = \frac{\int_0^4 f(x)J_0(\sqrt{\lambda_n}x)x\,dx}{\int_0^4 J_0^2(\sqrt{\lambda_n}x)x\,dx}.$$



Above is a graph showing the Fourier-Bessel approximation to the function with 50 terms in blue along with the points of convergence to $f(x)$ in black. This is followed by the MatLab program to produce this graph.

```
% Fourier Bessel series 2
clear                % Clear previous definitions
figure(101)          % Assign figure number
clf                  % Clear previous figures
hold off             % Start with fresh graph

NptsX=1000;          % number of x pts
x = linspace(0,4,NptsX);
g=@(z) besselj(0,z);
bz(1) = fsolve(g,2.4);
lam(1)=bz(1)/4;      % sqrt(e.v.)
evlam(1)=lam(1)^2;   % 1st e.v.
for n=2:50
    bz(n)=fsolve(g,bz(n-1)+pi);
    lam(n)=bz(n)/4;      % sqrt(e.v.s)
    evlam(n)=lam(n)^2;   % n^{th} e.v.
end
f1 = zeros(1,NptsX);
for n=1:50
    lamn = lam(n);
    fdenom = @(x) x.*(besselj(0,lamn*x)).^2;
    bdenom(n) = integral(fdenom,0,4);
    fnum = @(x) 4*x.*(besselj(0,lamn*x));
    bnum(n) = integral(fnum,1,3);
    b(n) = bnum(n)/bdenom(n);    % Fourier coefficients
    fn = b(n)*besselj(0,lamn*x); % Fourier function(n)
    f1 = f1+fn;
end
plot(x,f1,'b-','LineWidth',1.5);
hold on
plot([1 3],[4 4],'k-','LineWidth',1.5);
plot([0 1],[0 0],'k-','LineWidth',1.5);
plot([3 4],[0 0],'k-','LineWidth',1.5);
plot([1 3],[2 2],'ko','MarkerSize',5,'MarkerFaceColor','k');
grid;
h = legend('Fourier Bessel series, $n = 50$',...
     '$f(x)$', 'Location','south');
set(h,'Interpreter','latex')
h.FontSize = 10;
xlim([0,4]);
ylim([-1 5]);
xlabel('$x$','FontSize',12,'interpreter','latex');
ylabel('$y$','FontSize',12,'interpreter','latex');
set(gca,'FontSize',12);         % Axis tick font size
print -depsc bessel_fourier2.eps
```