1. (8pts) A classic enzymatic negative feedback model satisfies the system:

$$\begin{aligned}\dot{x}_1 &= \frac{3}{1+0.2x_2} - 0.5x_1,\\ \dot{x}_2 &= 5x_1 - x_2,\end{aligned}$$

where x_1 is an enzyme and x_2 is the endproduct. The equilibrium satisfies:

$$\frac{3}{1+0.2x_{2e}} - 0.5x_{1e} = 0 \quad \text{and} \quad x_{2e} = 5x_{1e},$$

so

$$\frac{3}{1+x_{1e}} = 0.5x_{1e}, \quad \text{or} \quad x_{1e}^2 + x_{1e} - 6 = 0.$$

Thus, $x_{1e} = 2$ or -3. However, only the positive solution makes sense, so the positive equilibrium is $(x_{1e}, x_{2e}) = (2, 10)$.

The Jacobian for this system is:

$$J(x_1, x_2) = \begin{pmatrix} -0.5 & -\frac{0.6}{(1+0.2x_2)^2} \\ 5 & -1 \end{pmatrix}.$$

At the equilibrium, the Jacobian is:

$$J(2, 10) = \begin{pmatrix} -0.5 & -\frac{1}{15} \\ 5 & -1 \end{pmatrix}.$$

The characteristic equation satisfies:

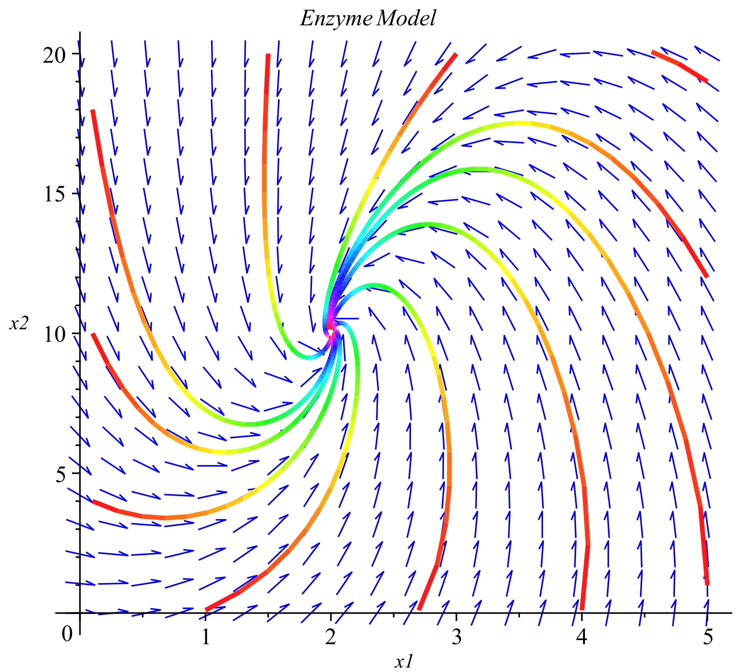
$$\begin{vmatrix} -\frac{1}{2} - \lambda & -\frac{1}{15} \\ 5 & -1-\lambda \end{vmatrix} = \lambda^2 + \frac{3}{2}\lambda + \frac{5}{6} = 0.$$

Thus, the eigenvalues at this equilibrium are:

$$\lambda = -\frac{3}{4} \pm \frac{i\sqrt{39}}{12} \approx -0.75 \pm 0.5204i,$$

which results in the local behavior near this equilibrium being a *stable spiral*.

Below is a phase portrait showing the trajectories of this system, spiraling toward the equilibrium at $(2, 10)$.



2. (8pts) The predator-prey model (Lotka-Volterra) is given by:

$$\begin{aligned}\dot{x}_1 &= 0.1x_1 - 0.05x_1x_2,\\ \dot{x}_2 &= 0.001x_1x_2 - 0.04x_2.\end{aligned}$$

We see that x_1 is the prey species, which grows according to Malthusian growth in the absence of the predator x_2. The prey species is lost due to interaction with the predator, shown by the product term $0.05x_1x_2$. The predator needs the prey species to reproduce and grow, which is given by the term $0.001x_1x_2$, while it dies off (radioactive decay–like term) proportional to its population.

The equilibria satisfy:

$$0.05x_{1e}(2 - x_{2e}) = 0 \quad \text{and} \quad 0.001x_{2e}(x_{1e} - 40) = 0.$$

From the first equation, if $x_{1e} = 0$, then the second equation implies $x_{2e} = 0$. From the first equation, if $x_{2e} = 2$, then the second equation implies $x_{1e} = 40$. Thus, there are two equilibria, the extinction and coexistence equilibria:

$$(x_{1e}, x_{2e}) = (0, 0) \qquad \text{and} \qquad (40, 2).$$

The Jacobian for this system is:

$$J(x_1, x_2) = \begin{pmatrix} 0.1 - 0.05x_2 & -0.05x_1 \\ 0.001x_2 & 0.001x_1 - 0.04 \end{pmatrix}.$$

For the extinction equilibrium, $(0, 0)$, the Jacobian is:

$$J(0, 0) = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.04 \end{pmatrix},$$

which is a diagonal matrix with eigenvalues, $\lambda_1 = 0.1$ and $\lambda_2 = -0.04$. Thus, there is a *saddle point* at this equilibrium.

For the cooperative equilibrium, $(40, 2)$, the Jacobian is:

$$J(40, 2) = \begin{pmatrix} 0 & -2 \\ 0.002 & 0 \end{pmatrix},$$

which has the characteristic equation:

$$\lambda^2 + 0.004 = 0, \qquad \text{so} \quad \lambda = \pm i\sqrt{0.004} \approx \pm 0.063246i.$$

Thus, there is a *center* at this equilibrium.

Below is a phase portrait showing the trajectories of this system, which are periodic orbits about the coexistence equilibrium. The trajectories along the axes show the saddle node at the extinction equilibrium. The violet dashed lines show the nullclines for $\dot{x}_1 = 0$, while the orange dashed lines show the nullclines for $\dot{x}_2 = 0$.