1. (5pts) For the linear system,

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 10 \end{pmatrix},$$

we first find the eigenvalues and eigenvectors by solving:

$$\begin{vmatrix} \lambda & 1 \\ -4 & 5-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0.$$

This is a companion matrix, so $\lambda_1 = 1$ has the corresponding eigenvector $\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Similarly, $\lambda_2 = 4$ has the corresponding eigenvector $\xi_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. It follows that the general solution satisfies:

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t}.$$

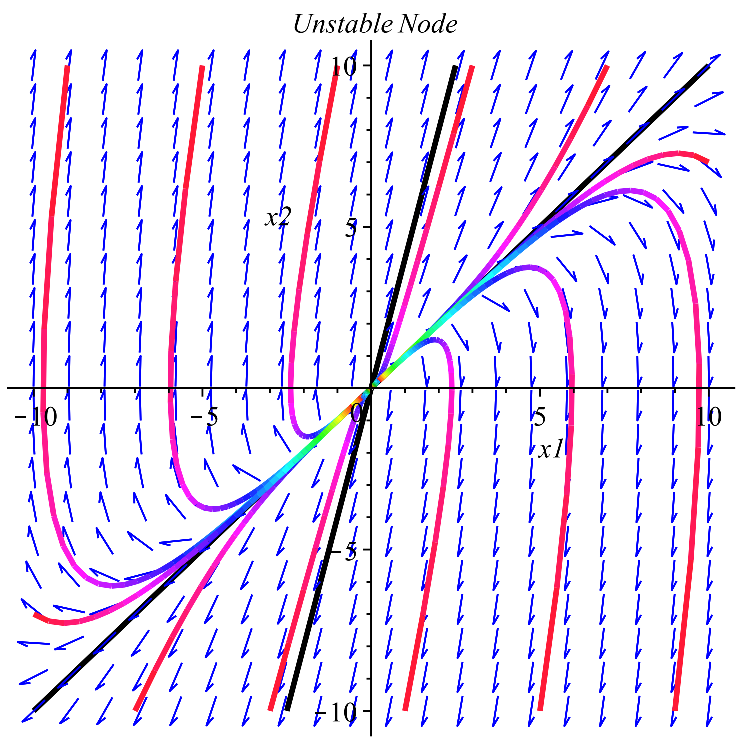
To satisfy the initial conditions, we solve:

$$\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix},$$

which is readily solved to give $c_1 = -2$ and $c_2 = 3$. Thus, the unique solution to the initial value is given by:

$$x(t) = \begin{pmatrix} -2 \\ -2 \end{pmatrix} e^t + \begin{pmatrix} 3 \\ 12 \end{pmatrix} e^{4t}.$$

Since both values of λ are positive, this is an unstable node. Below is a phase portrait showing the trajectories of this system, where the eigenvectors are shown in black.



2. (5pts) For the linear system,

$$\dot{\mathbf{x}} = \begin{pmatrix} -3 & 5 \\ -2 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} -2 \\ 2 \end{pmatrix},$$

we first find the eigenvalues and eigenvectors by solving:

$$\begin{vmatrix} -3-\lambda & 5 \\ -2 & -1-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 13 = (\lambda+2)^2 + 9 = 0,$$

which has eigenvalues, $\lambda = -2 \pm 3i$. For $\lambda = -2 + 3i$, we solve:

$$A - \lambda I = \begin{pmatrix} -1-3i & 5 \\ -2 & 1-3i \end{pmatrix} \xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{so} \quad \xi_1 = \begin{pmatrix} 5 \\ 1+3i \end{pmatrix} \quad \left(\text{or} \quad \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} \right).$$

It follows that

$$x_1(t) = e^{-2t} \begin{pmatrix} 5 \\ 1+3i \end{pmatrix} \big(\cos(3t) + i\sin(3t)\big) = e^{-2t} \left[\begin{pmatrix} 5\cos(3t) \\ \cos(3t) - 3\sin(3t) \end{pmatrix} + i \begin{pmatrix} 5\sin(3t) \\ \sin(3t) + 3\cos(3t) \end{pmatrix} \right].$$

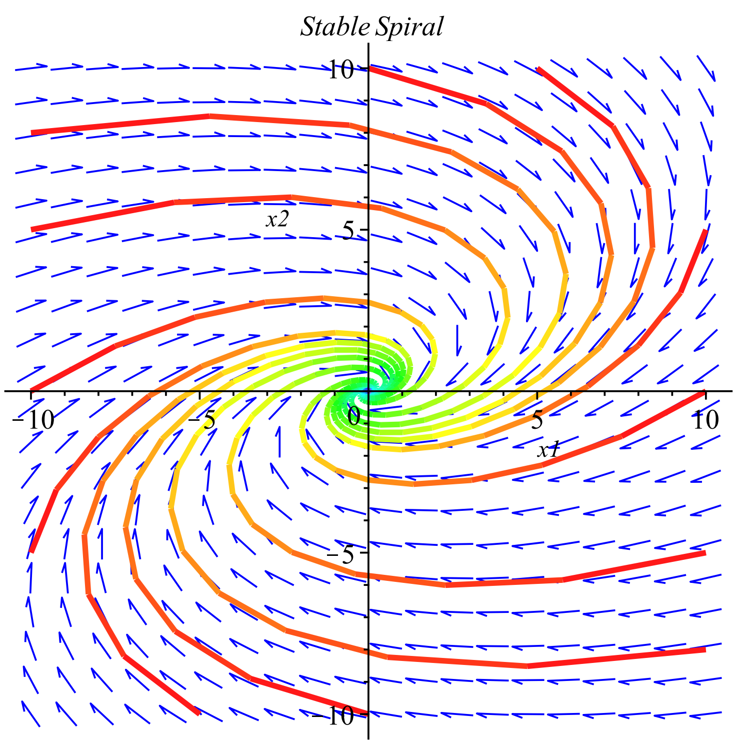
The general real solution from the real and imaginary parts satisfies:

$$x(t) = e^{-2t} \left[c_1 \begin{pmatrix} 5\cos(3t) \\ \cos(3t) - 3\sin(3t) \end{pmatrix} + c_2 \begin{pmatrix} 5\sin(3t) \\ \sin(3t) + 3\cos(3t) \end{pmatrix} \right].$$

From the initial conditions, we have $5c_1 = -2$ or $c_1 = -\frac{2}{5}$. Also, $c_1 + 3c_2 = 2$ or $c_2 = \frac{4}{5}$. The unique solution to this IVP becomes:

$$x(t) = e^{-2t} \begin{pmatrix} -2\cos(3t) + 4\sin(3t) \\ 2\cos(3t) + 2\sin(3t) \end{pmatrix}.$$

Since eigenvalues λ are complex with a negative real value, this is a stable spiral (clockwise). Below is a phase portrait showing the trajectories of this system.



3. (6pts) For the linear system,

$$\begin{aligned}\frac{dx_1}{dt} &= -2\,x_1 + 4\,x_2 + 2,\\ \frac{dx_2}{dt} &= x_1 + x_2 - 4,\end{aligned}$$

we find equilibria by solving:

$$\begin{pmatrix}-2 & 4\\ 1 & 1\end{pmatrix}\begin{pmatrix}x_{1e}\\ x_{2e}\end{pmatrix} = \begin{pmatrix}-2\\ 4\end{pmatrix}.$$

Adding 2 times the 2^{nd} row to the first gives, $6x_{2e} = 6$ or $x_{2e} = 1$. It follows that $x_{1e} = 3$, so $(x_{1e}, x_{2e}) = (3, 1)$ is the equilibrium.

Make the change of variables, $z_1 = x_1 - x_{1e}$ and $z_2 = x_2 - x_{2e}$, then the new system becomes:

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix} = \begin{pmatrix}-2 & 4\\ 1 & 1\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}.$$

We find the eigenvalues and eigenvectors by solving:

$$\begin{vmatrix}-2-\lambda & 4\\ 1 & 1-\lambda\end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0,$$

which has eigenvalues, $\lambda_1 = -3$ and $\lambda_2 = 2$. For $\lambda_1 = -3$, we solve:

$$A + 3I = \begin{pmatrix}1 & 4\\ 1 & 4\end{pmatrix}\xi_1 = \begin{pmatrix}0\\ 0\end{pmatrix}, \qquad \text{so} \quad \xi_1 = \begin{pmatrix}4\\ -1\end{pmatrix}.$$

Similarly, for $\lambda_2 = 2$, we solve:

$$A - 2I = \begin{pmatrix}-4 & 4\\ 1 & -1\end{pmatrix}\xi_2 = \begin{pmatrix}0\\ 0\end{pmatrix}, \qquad \text{so} \quad \xi_2 = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

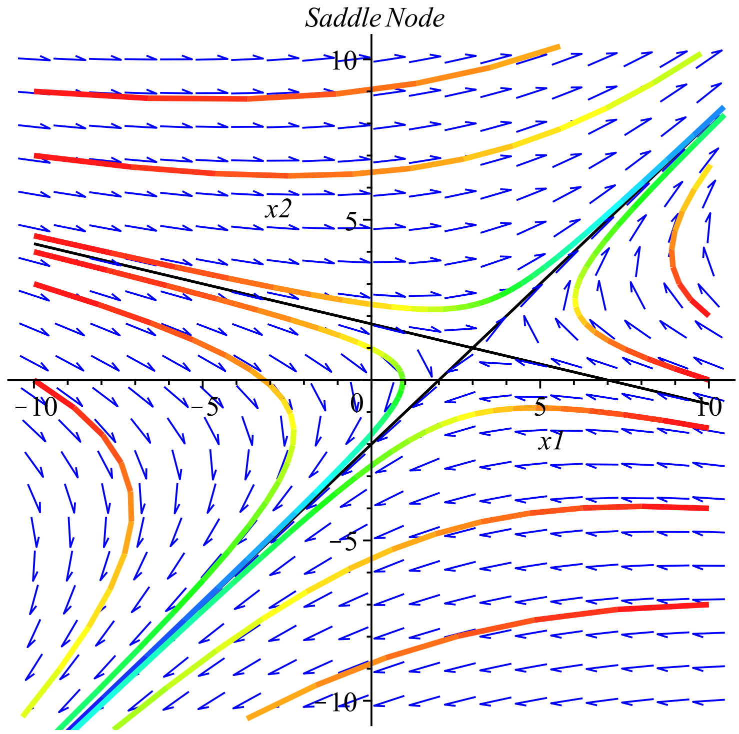
It follows that the general solution in the translated system satisfies:

$$z(t) = c_1\begin{pmatrix}4\\ -1\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}1\\ 1\end{pmatrix}e^{2t},$$

so

$$x(t) = c_1\begin{pmatrix}4\\ -1\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}1\\ 1\end{pmatrix}e^{2t} + \begin{pmatrix}3\\ 1\end{pmatrix}.$$

Since $\lambda_1 < 0 < \lambda_2$, this is an saddle node. Below is a phase portrait showing the trajectories of this system, where the eigenvectors are shown in black. The eigenvectors intersect at the equilibrium point.


Saddle Node
x2
x1
10
5
0
−5
−10
−10
−5
5
10