1. (4pts) The ODE is:

$$\frac{dy}{dt} = 3\,y - 20\,y^{\,2}e^{2t} \qquad \text{or} \qquad \frac{dy}{dt} - 3\,y = -20\,y^{\,2}e^{2t},$$

which is a Bernoulli equation. Thus, we substitute $u = y^{1-2} = y^{-1}$, so $\frac{du}{dt} = -y^{-2}\frac{dy}{dt}$. Multiplying the original ODE by $-y^{-2}$ gives:

$$\frac{du}{dt} + 3u = 20e^{2t},$$

which is a linear DE with integrating factor $\mu(t) = e^{3t}$. It follows that

$$\frac{d}{dt}\left(e^{3t}u\right) = 20e^{5t},$$

so integrating both sides gives.

$$e^{3t}u(t) = 4e^{5t} + C \qquad \text{or} \qquad u(t) = 4e^{2t} + Ce^{-3t}.$$

The initial condition is $y(0) = 2$, so

$$u(0) = \frac{1}{2} = 4 + C \qquad \text{or} \qquad C = -\frac{7}{2}.$$

It follows that:

$$u(t) = \frac{1}{y(t)} = 4e^{2t} - \frac{7}{2}e^{-3t},$$

or

$$y = \frac{2}{8e^{2t} - 7e^{-3t}}.$$

2. (4pts) The ODE is:

$$2y\Big(4 - \sin(3t)\Big)\frac{dy}{dt} = 3y^{\,2}\cos(3t) - 8t^{\,3}, \qquad \text{or} \qquad 8t^{\,3} - 3y^{\,2}\cos(3t) + 2y\Big(4 - \sin(3t)\Big)\frac{dy}{dt} = 0.$$

Checking

$$M_y = 6y\cos(3t) = N_t,$$

which shows this is an exact ODE, so we find this comes from a function $\phi(t, y)$. We integrate $M(t, y)$ with respect to t, giving

$$\phi(t, y) = \int \Big(8t^{\,3} - 3y^{\,2}\cos(3t)\Big)\,dt = 2t^{\,4} - y^{\,2}\sin(3t) + h(y),$$

and integrating $N(t, y)$ with respect to y gives,

$$\phi(t, y) = \int 2y\Big(4 - \sin(3t)\Big)dy = y^2\Big(4 - \sin(3t)\Big) + k(t).$$

It follows that we can take $h(y) = 4y^2$ and $k(t) = 2t^{\,4}$, so

$$\phi(y, t) = y^2\Big(4 - \sin(3t)\Big) + 2t^4 = C \qquad \text{with} \qquad y(0) = 2.$$

Thus, $C = 16$, so solving for $y(t)$ gives

$$y = \sqrt{\frac{16 - 2t^4}{4 - \sin(3t)}}.$$

3. (8pts) a. With

$$\frac{dP}{dt} = rP, \qquad P(0) = 52.4,$$

the Malthusian growth law applied to the population of the United Kingdom gives:

$$P(t) = 52.4e^{rt}.$$

With $P(20) = 56.3$, it follows that $52.4e^{20r} = 56.3$ or

$$r = \frac{1}{20}\ln\left(\frac{56.3}{52.4}\right) = 0.003589.$$

Thus, the Malthusian growth model for the United Kingdom is:

$$P(t) = 52.4e^{0.003589t}.$$

b. By separation of variables, the modified Malthusian growth model

$$\frac{dP}{dt} = (a - bt)P \qquad \text{satisfies} \qquad \int \frac{dP}{P} = \ln(P) = \int (a - bt)dt = at - \frac{bt^2}{2} + C,$$

where $C = \ln(52.4)$. Thus, it readily follows that

$$P(t) = P_0 e^{at - \frac{bt^2}{2}} = 52.4e^{at - \frac{bt^2}{2}}.$$

Using the logarithmic form and evaluating at $t = 20$, we have

$$\ln\left(\frac{56.3}{52.4}\right) = 0.071787944 = 20a - 200b.$$

Similarly, at $t = 40$, we have

$$\ln\left(\frac{59.5}{52.4}\right) = 0.127069721 = 40a - 800b.$$

Taking 4 times the first equation minus the second equation gives

$$4\ln\left(\frac{56.3}{52.4}\right) - \ln\left(\frac{59.5}{52.4}\right) = 40a \qquad \text{or} \qquad a = 0.004002051.$$

Substituting and solving for b gives $b = 0.0000412654$. The modified Malthusian growth model for the United Kingdom satisfies:

$$P(t) = 52.4e^{0.0040021t - 0.000020633t^2}.$$

The model predicts that in 2020 with $t = 60$,

$$P(60) = 52.4e^{0.0040021(60) - 0.000020633(3600)} = 61.852.$$

The peak population occurs when $(a - bt)P = 0$, so $t = \frac{0.0040021}{0.000041265} = 96.983$ or just about 2057. At that time the model predicts that the population is $P(96.983) = 63.623$ million.