1. a.(4pts) According to this problem, where the tea cools according to Newton's law of cooling, we have:

$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(0) = 100 \qquad \text{and} \qquad T(2) = 95.$$

To solve this problem let $z(t) = T(t) - 20$, so $z(0) = 80$ and $z'(t) = T'(t)$. By substitution it follows that

$$\frac{dz}{dt} = -kz(t), \qquad z(0) = 80,$$

which has the solution

$$z(t) = 80e^{-kt} = T(t) - 20 \qquad \text{or} \qquad T(t) = 80e^{-kt} + 20.$$

At 2 minutes

$$T(2) = 95 = 80e^{-2k} + 20, \qquad \text{so} \qquad e^{2k} = \frac{80}{75} = \frac{16}{15}.$$

It follows that $k = \frac{1}{2}\ln\left(\frac{16}{15}\right) \approx 0.032269$.
After 5 minutes, $T(5) = 80e^{-0.032269(5)} + 20 = 88.08°\text{C}$, which when mixed with milk at $5°C$ gives:

$$T_m(5) = 88.08\left(\frac{4}{5}\right) + 5\left(\frac{1}{5}\right) = 71.46°\text{C}.$$

b. (4pts) If we immediately mix with the milk, then the initial condition becomes:

$$T_m(0) = 100\left(\frac{4}{5}\right) + 5\left(\frac{1}{5}\right) = 81°\text{C},$$

so

$$T_m(t) = 61e^{-kt} + 20.$$

With $k = 0.032269$ the cooling cup of tea satisfies the following at $t = 5$:

$$T_m(5) = 61e^{-0.032269(5)} + 20 = 71.91°\text{C}.$$

Thus, after 5 minutes the tea is hotter if the milk is immediately poured in, but only by 0.45°C.

2. (4pts) We rewrite the ODE in the form:

$$\frac{dy}{dt} = 7 - 0.5y = -0.5(y - 14) = f(y).$$

With the substitution $z(t) = y(t) - 14$ and $z'(t) = y'(t)$, we obtain the ODE

$$z' = -0.5z.$$

If $y(t_0) = y_0$, so $z(t_0) = y_0 - 14$, then the solution of the z-equation is:

$$z(t) = (y_0 - 14)e^{-0.5(t-t_0)} = y(t) - 14.$$

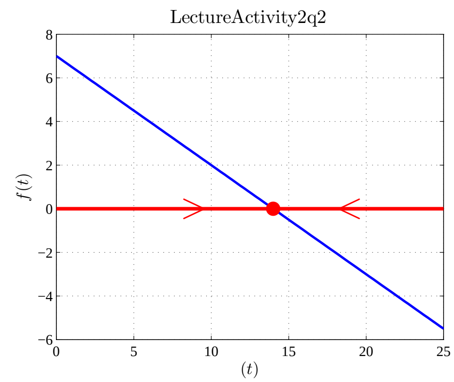
Thus, the general solution satisfies:

$$y(t) = 14 + (y_0 - 14)e^{-0.5(t-t_0)},$$

The equilibria satisfy $f(y_e) = 0$, so

$$-0.5(y_e - 14) = 0 \qquad \text{or} \qquad y_e = 14.$$

Since $f'(y) = -0.5 < 0$, then $f'(y_e) = -0.5 < 0$, which implies that $y_e = 14$ is an asymptotically stable equilibrium. Below is a graph showing the phase portrait of this ODE.



3. (4pts) The ODE is given by:

$$\frac{dy}{dt} = 0.5y^2 - 0.4y - 0.1y^3 = -0.1y(y^2 - 5y + 4) = f(y).$$

Equilibria are found by solving $f(y_e) = 0$, so

$$f(y_e) = -0.1y_e(y_e^2 - 5y_e + 4) = -0.1y_e(y_e - 4)(y_e - 1) = 0.$$

It follows that the equilibria are $y_e = 0,\ 1,\ 4$. Taking the derivative, we find

$$f'(y) = y - 0.4 - 0.3y^2.$$

For $y_e = 0$, we have $f'(0) = -0.4 < 0$, which makes this equilibrium asymptotically stable. For $y_e = 1$, we have $f'(1) = 0.3 > 0$, which makes this equilibrium unstable. For $y_e = 4$, we have $f'(4) = -1.2 < 0$, which makes this equilibrium asymptotically stable. The phase portrait showing the stability of the equilibria is below.