1. (4pts) For the 2^{nd} order linear nonhomogeneous ODE given by:

$$y'' - y' - 2y = 54t\,e^{2t} - 20t,$$

we begin by examining the homogeneous equation. The characteristic equation satisfies, $\lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2) = 0$ or $\lambda_1 = -1$ and $\lambda_2 = 2$. This gives the solution to the homogeneous equation as:

$$y(t) = c_1 e^{-t} + c_2 e^{2t}.$$

The right hand side of the nonhomogeneous equation has a linear polynomial times e^{2t} and a linear polynomial, so by the Table on Slide 24, we guess a particular solution of the form:

$$y_p(t) = t(A_1 t + A_0)e^{2t} + (B_1 t + B_0),$$

where the additional t is multiplying the first expression, so that it does **NOT** match any term in the homogeneous solution. Next find $y_p{}'$ and $y_p{}''$, so

$$\begin{aligned} y_p{}' &= \big((2t^2 + 2t)A_1 + (2t+1)A_0\big)e^{2t} + B_1, \\ y_p{}'' &= \big((4t^2 + 8t + 2)A_1 + (4t+4)A_0\big)e^{2t}. \end{aligned}$$

This is substituted into the original ODE to give:

$$\big((4t^2+8t+2)A_1 + (4t+4)A_0\big)e^{2t} - \big((2t^2+2t)A_1 + (2t+1)A_0\big)e^{2t} + B_1 - 2\big(t(A_1 t + A_0)e^{2t} + (B_1 t + B_0)\big) = 54t\,e^{2t} - 20t.$$

This simplifies to:

$$\big((6t+2)A_1 + 3A_0\big)e^{2t} - (2t+1)B_1 - 2B_0 = 54t\,e^{2t} - 20t.$$

Next we equate coefficients, so

$$\begin{array}{lll} te^{2t}: & 6A_1 = 54, \quad \text{or} \quad A_1 = 9, \\ e^{2t}: & 2A_1 + 3A_0 = 0, \quad \text{or} \quad A_0 = -6, \\ t: & -2B_1 = -20, \quad \text{or} \quad B_1 = 10, \\ 1: & -B_1 - 2B_0 = 0, \quad \text{or} \quad B_0 = -5. \end{array}$$

It follows that the particular solution is $y_p(t) = (9t^2 - 6t)e^{2t} + 10t - 5$, so the general solution is given by:

$$y(t) = c_1 e^{-t} + c_2 e^{2t} + (9t^2 - 6t)e^{2t} + 10t - 5.$$

2. (4pts) For the 2^{nd} order linear nonhomogeneous ODE given by:

$$y'' + 4y' + 4y = 24te^{-2t} + 40\cos(2t),$$

we begin by examining the homogeneous equation. The characteristic equation is $\lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0$, which gives the homogeneous solution:

$$y_h(t) = c_1 e^{-2t} + c_2 te^{-2t}.$$

The Method of Undetermined Coefficients based on the RHS of the ODE above suggests the following particular solution:

$$y_p(t) = A_0\cos(2t) + B_0\sin(2t) + t^2(C_1t + C_0)e^{-2t}.$$

We differentiate this particular solution twice giving:

$$\begin{aligned} y_p'(t) &= -2A_0\sin(2t) + 2B_0\cos(2t) + \left(-2C_1t^3 + (3C_1 - 2C_0)t^2 + 2C_0t\right)e^{-2t}, \\ y_p''(t) &= -4A_0\cos(2t) - 4B_0\sin(2t) + \left(4C_1t^3 + (4C_0 - 12C_1)t^2 + (6C_1 - 8C_0)t + 2C_0\right)e^{-2t}. \end{aligned}$$

Substituting the trig parts back in the original equation gives,

$$-4A_0\cos(2t) - 4B_0\sin(2t) + 4(-2A_0\sin(2t) + 2B_0\cos(2t)) + 4(A_0\cos(2t) + B_0\sin(2t)) = 40\cos(2t).$$

This simplifies to

$$8B_0\cos(2t) - 8A_0\sin(2t) = 40\cos(2t),$$

which implies $B_0 = 5$ and $A_0 = 0$. Substituting the exponential parts back in the original equation gives,

$$\begin{aligned} \left(4C_1t^3 + (4C_0 - 12C_1)t^2 + (6C_1 - 8C_0)t + 2C_0\right)e^{-2t} + \\ 4\left(-2C_1t^3 + (3C_1 - 2C_0)t^2 + 2C_0t\right)e^{-2t} + 4(C_1t^3 + C_0t^2)e^{-2t} &= 24te^{-2t}. \end{aligned}$$

It is readily seen that the t^3e^{-2t} and t^2e^{-2t} cancel, leaving only

$$\left((6C_1 - 8C_0)t + 2C_0\right)e^{-2t} + 8C_0te^{-2t} = 24te^{-2t}.$$

The coefficient of e^{-2t} gives $C_0 = 0$, so $6C_1 = 24$ or $C_1 = 4$. The particular solution is combined with the homogeneous solution to give the general solution:

$$y(t) = c_1e^{-2t} + c_2te^{-2t} + 5\sin(2t) + 4t^3e^{-2t}.$$

3. (4pts) For the 2^{nd} order linear nonhomogeneous ODE given by:

$$y'' + 2y' + 4y = 8te^{-2t} + 12t^2.$$

we begin by examining the homogeneous equation. The characteristic equation is $\lambda^2 + 2\lambda + 4 = (\lambda + 1)^2 + 3 = 0$ or $\lambda = -1 \pm i\sqrt{3}$, which gives the homogeneous solution:

$$y_h(t) = e^{-t}\left(c_1\cos(\sqrt{3}t) + c_2\sin(\sqrt{3}t)\right).$$

The Method of Undetermined Coefficients based on the RHS of the ODE above suggests the following particular solution:

$$y_p(t) = (A_1t + A_0)e^{-2t} + B_2t^2 + B_1t + B_0.$$

We differentiate this particular solution twice giving:

$$\begin{aligned} y'_p(t) &= -2(A_1t + A_0)e^{-2t} + A_1e^{-2t} + 2B_2t + B_1 \;=\; (-2A_1t + A_1 - 2A_0)e^{-2t} + 2B_2t + B_1, \\ y''_p(t) &= -2(-2A_1t + A_1 + A_0)e^{-2t} - 2A_1e^{-2t} + 2B_2 \;=\; (4A_1t - 4A_1 - 2A_0)e^{-2t} + 2B_2. \end{aligned}$$

Substituting y_p back in the original equation gives,

$$\begin{aligned} &(4A_1t - 4A_1 - 2A_0)e^{-2t} + 2B_2 + 2((-2A_1t + A_1 + A_0)e^{-2t} + 2B_2t + B_1) \\ &\qquad\qquad 4((A_1t + A_0)e^{-2t} + B_2t^2 + B_1t + B_0) \;=\; 8te^{-2t} + 12t^2. \end{aligned}$$

Next we equate coefficients, so

$$\begin{array}{rl|rl} te^{-2t}: & 4A_1 - 4A_1 + 4A_1 = 8, & t^2: & 4B_2 = 12, \\ e^{-2t}: & -4A_1 - 2A_0 + 2A_1 + 2A_0 + 4A_0 = 0, & t: & 4B_2 + 4B_1 = 0, \\ & & t^0: & 2B_2 + 2B_1 + 4B_0 = 0. \end{array}$$

From te^{-2t}, we have $A_1 = 2$, so from e^{-2t}, we have $A_0 = 1$. The cascade from the powers of t yield $B_2 = 3$, $B_1 = -3$, and $B_0 = 0$. Thus, it follows that:

$$y_p(t) = (2t + 1)e^{-2t} + 3t^2 - 3t,$$

so the general solution is:

$$y(t) = e^{-t}\left(c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)\right) + (2t + 1)e^{-2t} + 3t^2 - 3t.$$

4. (4pts) For the following nonhomogeneous differential equation we give the form of the particular solution that you would guess in using the **method of undetermined coefficients**. We (**DO NOT** solve for the undetermined coefficients.) For

$$y'' - 2y' + y = 5t\,e^t \sin(2t) + 20t^2e^t,$$

We examine the homogeneous equation. The characteristic equation is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$, which gives the homogeneous solution:

$$y_h(t) = c_1e^t + c_2te^t.$$

The Method of Undetermined Coefficients based on the RHS of the ODE above suggests the following particular solution:

$$y_p(t) = e^t\left((A_1t + A_0)\sin(2t) + (B_1t + B_0)\cos(2t)\right) + t^2(C_2t^2 + C_1t + C_0)e^t.$$