This Lecture Activity is designed to have you actively work with the lecture notes presented in class and available on my website. This activity is meant to keep you engaged and current with the class, so there is a fairly rapid turn around (due by **Thur. Mar 10 by noon**). There is an application (several parts) that require written answers, which are entered into **Gradescope**.

Note: For full credit you must show intermediate steps in your calculations.

This problem is closely related to the Greenhouse example given in class, so you will follow similar steps to the Lecture Notes (LinSys2A). The example below is characteristic of many two-compartment models with external input.

1. (1pts) a. The model below represents a situation of two distinct variables, $x_1(t)$ and $x_2(t)$, interacting linearly with each other like the temperatures of the air in the Greenhouse and the air above the Rock bed. You could also think of two distinct chemical elements, pharmaceutical drugs, or biological species that mix in a linear manner that results in a dynamic change of their state. Assume that measurements lead to the following two ODEs:

$$\begin{aligned} \dot{x}_1 &= \tfrac{13}{10}x_2 - \tfrac{7}{5}x_1 - \tfrac{6}{5}, \\ \dot{x}_2 &= \tfrac{1}{10}x_1 - \tfrac{1}{5}x_2 + \tfrac{9}{5}. \end{aligned} \tag{1}$$

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, A be a 2×2 matrix, and $\mathbf{b}$ be a 2×1 vector. Write the two ODEs (1) into a matrix system of the form (Slides LinSysA 9–10):

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}.$$

b. (2pts) The next step in analysis of this system is finding any equilibria for the system. For this analysis, we solve the linear system, $\dot{\mathbf{x}} = \mathbf{0}$ or

$$A\mathbf{x}_e + \mathbf{b} = \mathbf{0}.$$

Find $\mathbf{x}_e = \begin{pmatrix} x_{1e} \\ x_{2e} \end{pmatrix}$ that satisfies this linear system (Slides LinSysA 12–13).

c. (2pts) Next make a change of variables $\mathbf{y} = \mathbf{x} - \mathbf{x}_e$, creating the homogeneous linear system,

$$\dot{\mathbf{y}} = A\mathbf{y}. \tag{2}$$

What is the equilibrium for this system, $\mathbf{y}_e$ (Slide LinSysA 15)?

d. (4pts) Attempt a solution of the form, $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$, and substitute this into (2). Find the appropriate eigenvalue problem and solve the eigenvalue problem, finding the eigenvalues and eigenvectors (Slides LinSysA 16–20).

e. (4pts) Write the general solution, $\mathbf{y}(t)$, with the two linearly independent solutions you found above, using arbitrary constants, c_1 and c_2. Transform the solution back to $\mathbf{x}(t)$ with your equilibrium solution. Then consider the initial conditions (ICs):

$$\mathbf{x}(0) = \begin{pmatrix} 6 \\ 24 \end{pmatrix},$$

and find the unique solution to the IVP for (1) with these ICs (Slides LinSysA 21–22).

f. (3pts) Use MatLab to graph the solutions, $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, for $t \in [0, 20]$. Note any horizontal asymptotes for these solutions (Slide LinSysA 23).