1. For the following 1^{st} order systems of differential equations, find the general solution and determine the specific solution to the initial value problem. Sketch the phase portrait for typical solutions, including the specific solution to the initial value problem. When the eigenvectors are real, show the eigenvectors in the phase portrait. For complex eigenvalues show if trajectories are clockwise or counterclockwise. State the type of node at your equilibrium.

a. $\dot{\mathbf{x}} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

b. $\dot{\mathbf{x}} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

c. $\dot{\mathbf{x}} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$

d. $\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

e. $\dot{\mathbf{x}} = \begin{pmatrix} -1 & -\frac{1}{2} \\ 2 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

f. $\dot{\mathbf{x}} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

2. Consider the 1^{st} order system of differential equations given by:

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix}\mathbf{x},$$

where α is a parameter. Find the characteristic equation and eigenvalues in terms of α. Determine the critical values of α where the qualitative nature of the phase portrait changes, giving information on the changes that occur. Sketch a phase portrait for values of α slightly above and below the critical values.

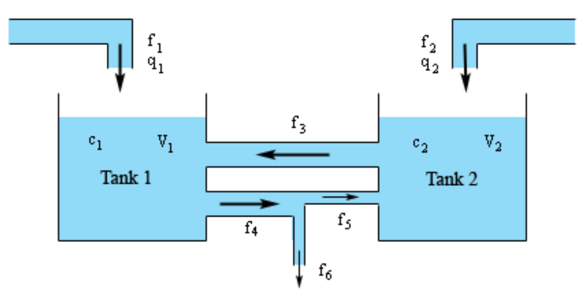
3. a. Consider the following initial value problem:

$$\begin{aligned} \frac{dx_1}{dt} &= -x_1 - 4x_2 + 6, \qquad x_1(0) = 5, \\ \frac{dx_2}{dt} &= x_1 - x_2 + 4, \qquad x_2(0) = 2. \end{aligned}$$

Find the general solution and the unique solution to the IVP.

b. Sketch the phase portrait for typical solutions, including the specific solution to the initial value problem. When the eigenvectors are real, show the eigenvectors in the phase portrait. For complex eigenvalues show if trajectories are clockwise or counterclockwise. State the type of node at your equilibrium.

4. This problem examines the mixing of an **inert salt** in two tanks. The figure on the right shows two tanks, which are connected and having saline solutions flowing into them, between them, and out of them. They begin with initial concentrations of $c_1(0) = 2$ g/l and $c_2(0) = 3$ g/l. The flow rates balance so that they maintain



constant volumes, $V_1 = 200$ l, and $V_2 = 100$ l. The concentration of the brine flowing into Tank 1 is $q_1 = 8$ g/l, while the concentration of the brine flowing into Tank 2 is $q_2 = 15$ g/l. The flow rates are given below:

$$\begin{array}{lll} f_1 = 0.3 \text{ l/min}, & f_2 = 0.2 \text{ l/min}, & f_3 = 0.4 \text{ l/min}, \\ f_4 = 0.7 \text{ l/min}, & f_5 = 0.2 \text{ l/min}, & f_6 = 0.5 \text{ l/min} \end{array}$$

a. Set up the differential equation describing the concentrations, $c_1(t)$ and $c_2(t)$, for this system. Solve this system of differential equations (both general and specific solutions) and describe what happens for large time.

b. Sketch the phase portrait for typical solutions, including the specific solution to the initial value problem. When the eigenvectors are real, show the eigenvectors in the phase portrait. For complex eigenvalues show if trajectories are clockwise or counterclockwise. State the type of node at your equilibrium.

5. a. Prey animals often satisfy logistic growth dynamics. Predators and prey interact in a manner where the contact between predators and prey can result in the loss of the prey with a gain of energy for the predator. Let the density of prey be $H(t)$ and the density of predators be $P(t)$. Assume that this pair of predators and prey are tightly linked with each other. Suppose that the dynamics of this interaction satisfy the following system of equations:

$$\begin{aligned} \frac{dH}{dt} &= 0.1H - 0.0005H^2 - 0.016HP, \\ \frac{dP}{dt} &= 0.005HP - 0.2P. \end{aligned}$$

Find all equilibria. Linearize the model around each of the equilibria. Find the eigenvalues and eigenvectors at each of the equilibria and determine what type of node is occurring.

b. Create a diagram in phase space, showing clearly the equilibria and **all** nullclines ($\dot{x} = 0$ or $\dot{y} = 0$). Sketch a typical solution starting from $(x(0), y(0)) = (2, 1)$.

6. a. Species often compete for the same resources. Ecologically, the more species there are in an ecosystem, the more stable the system is. When there are only two species, then the resulting competition usually leads to either **competitive exclusion** or **coexistence** depending how the species interact with their own kind and each other. Consider a two species competition model with the species $x_1(t)$ and $x_2(t)$. Assume that the model satisfies the system of differential equations given by:

$$\begin{aligned} \frac{dx_1}{dt} &= 0.3x_1 - 0.005x_1^2 - 0.009x_1x_2, \\ \frac{dx_2}{dt} &= 0.1x_2 - 0.0025x_2^2 - 0.002x_1x_2. \end{aligned}$$

Find all equilibria. Linearize the model around each of the equilibria. Find the eigenvalues and eigenvectors at each of the equilibria and determine what type of node is occurring.

b. Create a diagram in phase space, showing clearly the equilibria and **all** nullclines ($\dot{x} = 0$ or $\dot{y} = 0$). Sketch typical solutions from the **4** distinct regions separated by the nullclines. Determine if this system results in **competitive exclusion** or **coexistence**.