1. a. This is a linear differential equation,so it can be written

$$\frac{dy}{dt} + (0.2t - 2)y = 0, \qquad \text{with} \qquad \mu(t) = e^{\int (0.2t-2)dt} = e^{0.1t^2 - 2t},$$

where $\mu(t)$ is the integrating factor. It follows:

$$\frac{d}{dt}\left(e^{0.1t^2-2t}y\right) = 0 \qquad \text{or} \qquad e^{0.1t^2-2t}y(t) = C$$

It follows that $y(t) = Ce^{2\,t - 0.1\,t^2}$. The initial condition $y(0) = 10 = C$. Hence, the solution is

$$y(t) = 10\,e^{2\,t-0.1\,t^2}.$$

b. This is a time varying differential equation. It can be written

$$y(t) = \int \left(2 - \frac{4}{t}\right) dt = 2\,t - 4\,\ln(t) + C.$$

The initial condition $y(1) = 5 = 2 + C$, which implies $C = 3$. Hence, the solution is $y(t) = 2\,t - 4\,\ln(t) + 3$.

c. This is a separable differential equation. It can be written

$$\int 2y\,dy = \int 3t^2\,dt \qquad \text{or} \qquad y^2(t) = t^3 + C.$$

It follows that $y(t) = \pm\sqrt{t^3 + C}$. The initial condition $y(0) = 4 = \sqrt{C}$, which implies $C = 16$. Hence, the solution is

$$y(t) = \sqrt{t^3 + 16}.$$

d. This is the logistic growth differential equation, which can be written

$$\frac{dy}{dt} = 0.02y\left(1 - \frac{y}{40}\right) \qquad \text{or} \qquad \frac{dy}{dt} - 0.02y = -0.0005y^2,$$

which is a Bernoulli's equation. Make the substitution $u = y^{1-2} = y^{-1}$, so $\frac{du}{dt} = -y^{-2}\frac{dy}{dt}$. Multiply the equation above by $-y^{-2}$, and

$$-y^{-2}\frac{dy}{dt} + 0.02y^{-1} = 0.0005 \qquad \text{or} \qquad \frac{du}{dt} + 0.02u = 0.0005,$$

which is a linear equation with integrating factor $\mu(t) = e^{0.02t}$. Thus,

$$\frac{d}{dt}\left(e^{0.02t}u\right) = 0.0005e^{0.02t} \qquad \text{or} \qquad e^{0.02t}u(t) = 0.025e^{0.02t} + C.$$

Hence, with the initial condition

$$\frac{1}{y(t)} = u(t) = 0.025 + Ce^{-0.02t} \quad \text{or} \quad 0.1 = 0.025 + C, \quad \text{so} \quad C = 0.075.$$

It follows that

$$y(t) = \frac{1}{0.025 + 0.075e^{-0.02t}} = \frac{40}{1 + 3e^{-0.02t}}.$$

e. Rewrite the equation as

$$3y - 6t + (3t + 4y)\frac{dy}{dt} = 0.$$

Since $\frac{\partial M(t,y)}{\partial y} = 3 = \frac{\partial N(t,y)}{\partial t}$, this equation is exact. Integrating we see

$$\int (3y - 6t)dt = 3ty - 3t^2 + h(y) \quad \text{and} \quad \int (3t + 4y)dy = 3ty + 2y^2 + k(t).$$

It is clear that the potential function is

$$\phi(t, y) = 3ty - 3t^2 + 2y^2 = C.$$

With the initial condition $y(0) = 4$, the solution becomes

$$\phi(t, y) = 3ty - 3t^2 + 2y^2 = 32.$$

f. This linear DE equation can be rewritten

$$\frac{dy}{dt} - \frac{2y}{t} = 4t^2 \sin(4t), \qquad \text{so} \qquad \mu(t) = e^{-\int 2dt/t} = \frac{1}{t^2}.$$

Thus,

$$\frac{d}{dt}\left(\frac{y}{t^2}\right) = 4\sin(4t) \qquad \text{or} \qquad \frac{y(t)}{t^2} = -\cos(4t) + C.$$

It follows that

$$y(t) = Ct^2 - t^2\cos(4t), \quad \text{so} \quad 2 = C - \cos(4) \quad \text{or} \quad C = 2 + \cos(4).$$

Hence, the solution is

$$y(t) = (2 + \cos(4))t^2 - t^2\cos(4t).$$

g. This is a linear and separable differential equation. We solve this time using separable techniques. The equation can be written

$$\int \frac{dy}{y} = \int \frac{2\,t\,dt}{t^2 + 1}.$$

The right integral uses the substitution $u = t^2 + 1$, so $du = 2\,t\,dt$. Hence,

$$\begin{aligned} \ln|y(t)| &= \int \frac{du}{u} = \ln|u| + C = \ln(t^2 + 1) + C \\ y(t) &= e^{\ln(t^2+1)+C} = A(t^2 + 1), \end{aligned}$$

where $A = e^C$. The initial condition $y(0) = 3 = A$, which implies $A = 3$. Hence, the solution is

$$y(t) = 3(t^2 + 1).$$

h. This is a separable differential equation. It can be written

$$\int e^y dy = \int e^t dt \qquad \text{or} \qquad e^y = e^t + C.$$

It follows that $y(t) = \ln(e^t + C)$. The initial condition $y(0) = 6 = \ln(1 + C)$, which implies $C = e^6 - 1$. Hence, the solution is

$$y(t) = \ln(e^t + e^6 - 1).$$

i. This is a Bernoulli equation, so rewrite

$$\frac{dy}{dt} - \frac{1}{t}y = 2y^2, \qquad \text{with} \qquad u = y^{1-2} = y^{-1} \quad \text{and} \quad \frac{du}{dt} = -y^{-2}\frac{dy}{dt}.$$

Thus,

$$-y^{-2}\left(\frac{dy}{dt} - \frac{1}{t}y = 2y^2\right) \qquad \text{becomes} \qquad \frac{du}{dt} + \frac{1}{t}u = -2,$$

which is a linear equation in u. It has the integrating factor $\mu(t) = e^{\int \frac{1}{t}dt} = t$, so

$$\frac{d(tu)}{dt} = -2t, \qquad \text{or} \qquad tu(t) = -2\int t\,dt + C = -t^2 + C.$$

With the initial condition, $y(1) = 1$, it follows that:

$$\frac{t}{y(t)} = -t^2 + C, \qquad \text{so} \quad 1 = -1 + C \quad \text{or} \quad C = 2.$$

The solution becomes:

$$y(t) = \frac{t}{2 - t^2}.$$

j. We rewrite the ODE:

$$\left(4t - \frac{2\cos(2t)}{y}\right) + \frac{(\sin(2t) - 2)}{y^2}\frac{dy}{dt} = 0.$$

We check for exactness:

$$\frac{\partial M}{\partial y} = \frac{2\cos(2t)}{y^2} = \frac{\partial N}{\partial t}.$$

First integrate $M(t, y)$ with respect to t, so

$$\phi(t, y) = \int \left(4t - \frac{2\cos(2t)}{y}\right) dt = 2t^2 - \frac{\sin(2t)}{y} + h(y).$$

Next integrate $N(t, y)$ with respect to y, so

$$\phi(t, y) = \int \frac{(\sin(2t) - 2)}{y^2} dy = -\frac{\sin(2t)}{y} + \frac{2}{y} + k(t).$$

Combining these results give:

$$\phi(t, y) = -\frac{\sin(2t)}{y} + \frac{2}{y} + 2t^2 = C \qquad \text{with} \qquad y(0) = 2.$$

The initial condition gives, $C = 1$, which allows solving for $y(t)$ giving:

$$y(t) = \frac{2 - \sin(2t)}{1 - 2t^2}.$$

k. Rewrite this equation:

$$ye^t - 2 + (e^t - 2y)\frac{dy}{dt} = 0.$$

Since $\frac{\partial M(t,y)}{\partial y} = e^t = \frac{\partial N(t,y)}{\partial t}$, this equation is exact. Integrating we see

$$\int (ye^t - 2)dt = ye^t - 2t + h(y) \quad \text{and} \quad \int (e^t - 2y)dy = ye^t - y^2 + k(t).$$

It is clear that the potential function is

$$\phi(t, y) = ye^t - 2t - y^2 = C.$$

With the initial condition $y(0) = 6$, the solution becomes

$$\phi(t, y) = y(t)e^t - 2t - y^2(t) = 6 - 36 = -30.$$

l. The DE

$$\frac{dy}{dt} + y = y^3 e^t$$

is a Bernoulli's equation, where we make the substitution $u = y^{1-3} = y^{-2}$, so $\frac{du}{dt} = -2y^{-3}\frac{dy}{dt}$. Multiplying the above equation by $-2y^{-3}$, we obtain the linear DE in $u(t)$

$$-2y^{-3}\frac{dy}{dt} - 2y^{-2} = -2e^t \qquad \text{or} \qquad \frac{du}{dt} - 2u = -2e^t.$$

This has the integrating factor $\mu(t) = e^{-2t}$, so

$$\frac{d}{dt}\left(e^{-2t}u(t)\right) = -2e^{-t} \qquad \text{or} \qquad e^{-2t}u(t) = 2e^{-t} + C.$$

It follows that

$$\frac{1}{y^2(t)} = u(t) = 2e^t + Ce^{2t}, \qquad \text{so} \qquad 1 = 2 + C \qquad \text{or} \qquad C = -1.$$

Thus,

$$y(t) = \frac{1}{\sqrt{2e^t - e^{2t}}}.$$

m. The linear ODE, $t\frac{dy}{dt} = 6t - 2y - 4$ with $y(1) = 3$, can be written

$$\frac{dy}{dt} + \frac{2}{t}y = 6 - \frac{4}{t}.$$

This has the integrating factor, $\mu(t) = e^{\int \frac{2}{t}\,dt} = t^2$, so

$$\frac{d}{dt}(t^2 y) = 6t^2 - 4t, \qquad \text{so} t^2 y(t) = \int \left(6t^2 - 4t\right)dt = 2t^3 - 2t^2 + C.$$

The initial condition implies, $3 = 2 - 2 + C = C$, so the solution is:

$$y(t) = 2t - 2 + \frac{3}{t^2}.$$

2. a. The solution to the white lead problem is $P(t) = 10e^{-kt}$, where $t = 0$ represents 1970. From the data at 1975, we have $8.5 = 10e^{-5k}$ or $e^{5k} = 10/8.5 = 1.17647$. Thus, $k = 0.032504$ yr^{-1}. To find the half-life, we compute $5 = 10e^{-kt}$, so $t = \ln(2)/k = 21.33$ yr is the half-life of lead-210.

b. The differential equation can be written $P' = -k(P - r/k)$, so we make the substitution $z(t) = P(t) - r/k$. This leaves the initial value problem

$$z' = -kz, \quad z(0) = P(0) - r/k = 10 - r/k,$$

which has the solution $z(t) = (P(0) - r/k)e^{-kt} = P(t) - r/k$. Thus, the solution is

$$P(t) = \left(10 - \frac{r}{k}\right) e^{-kt} + \frac{r}{k} = 2.3086e^{-kt} + 7.6914,$$

where $k = 0.032504$. In the limit,

$$\lim_{t\to\infty} P(t) = 7.6914 \text{ disintegrations per minute of } ^{210}\text{Pb}.$$

3. a. The differential equation describing the temperature of the tea satisfies

$$H' = -k(H - 21), \quad H(0) = 85 \text{ and } H(5) = 81.$$

Make the substitution $z(t) = H(t) - 21$, which gives the differential equation

$$z' = -kz, \quad z(0) = H(0) - 21 = 64.$$

The solution becomes $z(t) = 64e^{-kt} = H(t) - 21$ or

$$H(t) = 64e^{-kt} + 21.$$

To find k, we solve $H(5) = 81 = 64e^{-5k} + 21$ or $e^{5k} = 64/60 = 1.0667$. Thus, $k = 0.012908$ min^{-1}. The water was at boiling point when $64e^{-kt} + 21 = 100$ or $e^{-kt} = 79/64$. It follows that $t = -\ln(79/64)/k = -16.3$ min. This means that the talk went 16.3 min over its scheduled ending.

b. To obtain a temperature of at least 93°C, then we need to find the time that satisfies $H(t) = 93 = 64e^{-kt} + 21$, so $e^{-kt} = 72/64 = 1.125$. Solving for t gives $t = -\ln(72/64)/k = -9.125$ min. It follows that you must arrive at the hot water within $16.3 - 9.1 = 7.2$ min of the scheduled end of the talks.

4. a. In lecture, this type of pollution problem is solved using the change in amount equals the amount entering minus the amount leaving or $\frac{da}{dt} = Qf_1 - cf_2$, where $Q = 10$ is the concentration entering, c is the concentration in the lake, and $f_1 = 2200$ and $f_2 = 2000$ are the flows entering and leaving the lake. (Evaporation contributes nothing to amounts of pollutant.) Substituting the parameters into the differential equation, dividing by $V = 10^6$, and using $c(t) = \frac{a(t)}{V}$, gives

$$c' = \frac{1}{10^6}(22000 - 2000c) = -0.002(c - 11).$$

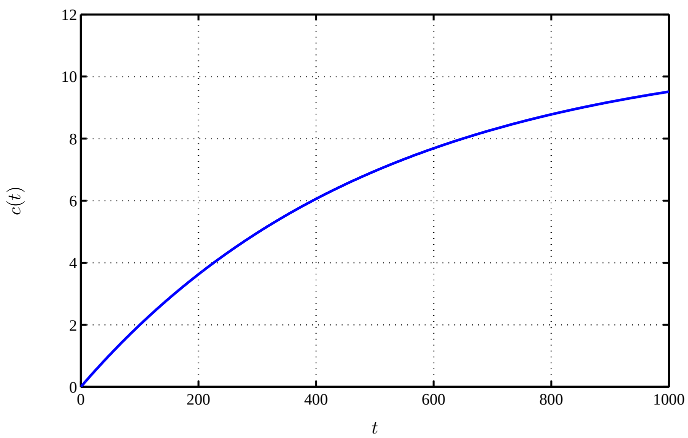
We make the substitution $z(t) = c(t) - 11$, which gives the initial value problem $z' = -0.002z$ with $z(0) = c(0) - 11 = -11$. The solution of this differential equation is $z(t) = -11e^{-0.002t} = c(t) - 11$, so

$$c(t) = 11 - 11e^{-0.002t}.$$

b. Solve the equation $c(t) = 11 - 11e^{-0.002t} = 5$, so $e^{0.002t} = 11/6$ or $t = 500\ln(11/6) = 303.1$ days. The limiting concentration

$$\lim_{t\to\infty} c(t) = 11.$$

The graph is below.



Problem 4

5. a. Fluoxetine, F, satisfies the ODE:

$$\frac{dF}{dt} = -0.5F, \qquad F(0) = 21 \text{ ng/ml}.$$

The solution to this ODE is:

$$F(t) = 21e^{-0.5t} \text{ ng/ml}.$$

The half life satisfies, $t_h = \frac{\ln(2)}{0.5} \approx 1.386$ days

b. Norfluoxetine, N, satisfies the ODE:

$$\frac{dN}{dt} = \frac{F(t)}{3} - 0.08N, \qquad N(0) = 0 \text{ ng/ml},$$

which is readily written as the linear ODE:

$$\frac{dN}{dt} + 0.08N = 7e^{-0.5t}, \quad \text{with integrating factor} \quad \mu(t) = e^{0.08t}.$$

Thus, we have

$$\frac{d}{dt}\left(e^{0.08t}N\right) = 7e^{-0.42t}, \qquad \text{or} \quad e^{0.08t}N = -\frac{7}{0.42}e^{-0.42t} + C.$$

With the initial condition, $N(0) = 0$, we have $C = \frac{7}{0.42}$, so

$$N(t) = \frac{7}{0.42}\left(e^{-0.08t} - e^{-0.5t}\right) = \frac{50}{3}\left(e^{-0.08t} - e^{-0.5t}\right) \text{ ng/ml.}$$

c. The maximum occurs when $N'(t) = 0$, so

$$N'(t) = \frac{50}{3}\left(-0.08e^{-0.08t} + 0.5e^{-0.5t}\right) = 0.$$

Equivalently,

$$0.08e^{-0.08t} = 0.5e^{-0.5t} \qquad \text{or} \qquad e^{0.42t} = \frac{0.5}{0.08} = \frac{25}{4}$$

It follows that

$$t_{max} = \frac{1}{0.42}\ln\left(\frac{25}{4}\right) \approx 4.3633 \text{ min.}$$

with $N(t_{max}) = 9.8749$ ng/ml

6. The differential equation with the information in the problem is given by:

$$\frac{dH}{dt} = -k(H - 25), \qquad H(0) = 35,$$

where $t = 0$ is 7 AM. We make the change of variables $z(t) = H(t) - 25$, so $z(0) = 10$. The problem now becomes

$$\frac{dz}{dt} = -kz, \qquad z(0) = 10,$$

which has the solution

$$z(t) = 10\,e^{-kt} \quad \text{or} \quad H(t) = 25 + 10\,e^{-kt}.$$

From the information at 9 AM, we see

$$H(2) = 33.5 = 25 + 10\,e^{-2k} \quad \text{or} \quad e^{2k} = \frac{10}{8.5} \quad \text{or} \quad k = \frac{\ln\left(\frac{10}{8.5}\right)}{2} = 0.081259.$$

It follows that

$$H(t) = 25 + 10\,e^{-0.081259t}.$$

The time of death is found by solving

$$H(t_d) = 39 = 25 + 10\,e^{-0.081259t_d} \quad \text{or} \quad e^{-0.081259t_d} = \frac{14}{10} \quad \text{or} \quad t_d = -\frac{\ln(1.4)}{0.081259} = -4.1407.$$

It follows that the time of death is 4 hours and 8.4 min before the body is found, which gives the time of death around 2:52 AM.

b. This differential equation is separable, so we can write:

$$\begin{aligned}
\int (H - 25)^{-2/3} dH &= -k_b \int dt = -k_b t + C, \\
3(H - 25)^{1/3} &= -k_b t + C, \\
H(t) &= 25 + \left(\frac{C - k_b t}{3}\right)^3.
\end{aligned}$$

The initial temperature of the body gives:

$$35 = 25 + \left(\frac{C}{3}\right)^3 \qquad \text{or} \qquad C = 3(10)^{1/3} \approx 6.4633.$$

From the temperature at $t = 2$,

$$33.5 = 25 + \left(10^{1/3} - \tfrac{2}{3}k_b\right)^3 \qquad \text{or} \qquad 8.5^{1/3} = 10^{1/3} - \tfrac{2}{3}k_b,$$

so

$$k_b = 1.5\left(10^{1/3} - 8^{1/3}\right) \approx 0.17041.$$

It follows that the time of death satisfies:

$$39 = 25 + \left(10^{1/3} - \tfrac{t_d}{3}k_b\right)^3 \qquad \text{or} \qquad 10^{1/3} - 14^{1/3} = \frac{t_d}{3}k_b.$$

Thus,

$$t_d = \frac{3}{k_b}\left(10^{1/3} - 14^{1/3}\right) \approx -4.5016 \qquad \text{or} \qquad 4 \text{ hr } 30.1 \text{ min},$$

which is approximately 2:29.9 AM. These models differ about 22 min in their predictions for the time of death.

7. a. The solution of the Malthusian growth model is $B(t) = 1000\,e^{0.01\,t}$. The population doubles when the bacteria reaches 2000, so $1000\,e^{0.01\,t} = 2000$ or $e^{0.01\,t} = 2$. Thus, $0.01\,t = \ln(2)$ or $t = 100\ln(2) \approx 69.3$ min for the population to double.

b. The model with time-varying growth is a linear and separable differential equation, so

$$\frac{dB}{dt} = 0.01(1 - e^{-t})\,B \quad \text{or} \quad \int \frac{dB}{B} = 0.01\int (1 - e^{-t})dt$$

$$\ln|B(t)| = 0.01(t + e^{-t}) + C \quad \text{or} \quad B(t) = A\,e^{0.01(t+e^{-t})},$$

where $A = e^C$. With the initial condition, $B(0) = 1000 = A\,e^{0.01}$ or $A = 1000\,e^{-0.01}$. Thus, the solution to this time-varying growth model is

$$B(t) = 1000\,e^{0.01(t+e^{-t}-1)}.$$

c. The Malthusian growth model gives $B(5) = 1051$ and $B(60) = 1822$, while the modified growth model gives $B(5) = 1041$ and $B(60) = 1804$.

8. a. The solution to the Malthusian growth model is given by $P(t) = 100\,e^{0.2\,t}$. This population doubles when $100\,e^{0.2\,t} = 200$ or $e^{0.2\,t} = 2$, so $t = 5\ln(2) \approx 3.466$ yrs.

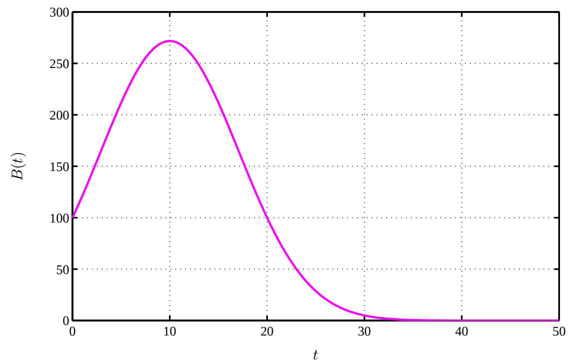
b. This model, including the modification for habitat encroachment, is a linear and separable differential equation. It can be written

$$\int \frac{dP}{P} = \int (0.2 - 0.02t)dt \qquad \text{or} \qquad \ln|P| = 0.2\,t - 0.01\,t^2 + C.$$

It follows that $P(t) = e^{0.2\,t-0.01\,t^2+C} = Ae^{0.2\,t-0.01\,t^2}$, where $A = e^C$. The initial condition $P(0) = 100 = A$, which implies $A = 100$. Hence, the solution satisfies

$$P(t) = 100\,e^{0.2\,t-0.01\,t^2}.$$

c. We examine the differential equation in Part b and see that $\frac{dP}{dt} = 0$ when $0.2 - 0.02t = 0$, which implies that $t = 10$. Thus, the maximum of population is $P(10) = 100\,e \approx 271.8$. If we solve $P(t) = 100\,e^{0.2\,t - 0.01\,t^2} = 100$, then this is equivalent to $e^{0.2\,t-0.01\,t^2} = 1$ or $0.2\,t - 0.01\,t^2 = -0.01\,t(t-20) = 0$. Thus, either $t = 20$ (or 0), so the population returns to 100 after 20 years. The graph of the population can be seen below.



9. a. From lecture notes, we write an ODE governing the amounts of pesticide in the lake. With the parameters, $V = 400,000$ m^3, $f_1 = 300$ m^3/day and $Q_1 = 12\,\mu$g/m^3, $f_2 = 500$ m^3/day and $Q_2 = 4\,\mu$g/m^3, and the assumption of well-mixed and constant volume, so $f_3 = 800$ m^3/day, we write the ODE in concentration via an ODE in amounts as follows:

$$\frac{dA}{dt} = 300 \cdot 12 + 500 \cdot 4 - 800c, \qquad \text{so} \qquad \frac{dc}{dt} = \frac{5600 - 800c}{400000} = -0.002(c-7).$$

We make the substitution, $z(t) = c(t) - 7$, which gives the simpler equation:

$$\frac{dz}{dt} = -0.002z, \quad z(0) = -7, \qquad \text{or} \qquad z(t) = -7e^{-0.002t}.$$

It follows that

$$c(t) = 7 - 7e^{-0.002t}.$$

b. The lake has a concentration of $4\,\mu$g/m^3 of pesticide when

$$7 - 7e^{-0.002t} = 4, \quad \text{or} \quad 3 = 7e^{-0.002t}, \quad \text{or} \quad e^{0.002t} = \frac{7}{3}, \quad \text{so} \quad t_2 = 500 \ln\left(\frac{7}{3}\right).$$

Thus, when $t_2 = 423.65$ days, we have $c(t_2) = 4$. The limiting concentration is when $\frac{dc}{dt} = 0$ or $c = 7\,\mu$g/m^3 of pesticide in this lake.

c. With the ODE for the population satisfying the following growth model:

$$\frac{dP}{dt} = (0.08 - 0.002\,t)P^{3/4}, \qquad P(0) = 1296,$$

we apply separation of variables, so:

$$\int P^{-\frac{3}{4}}\,dP = \int (0.08 - 0.002t)dt, \quad \text{or} \quad 4P^{\frac{1}{4}} = 0.08t - 0.001t^2 + C.$$

Thus,

$$P(t) = \left(0.02t - 0.00025t^2 + \frac{C}{4}\right)^4,$$

where the initial condition gives, $P(0) = \left(\frac{C}{4}\right)^4 = 1296$ or $\frac{C}{4} = 6$. It follows that

$$P(t) = \left(0.02t - 0.00025t^2 + 6\right)^4.$$

The maximum occurs when $\frac{dP}{dt} = 0$, so $0.08 = 0.002t$ or $t = 40$ days. Substituting the into the solution gives the maximum Population,$P(40) = 1677.72$. We find $P(100) = 915.06$.

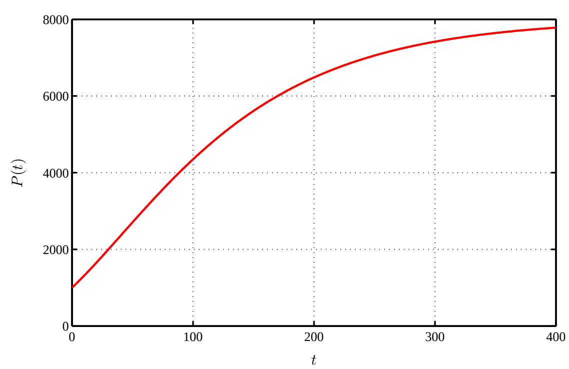
10. a. This population of cells in a declining medium satisfies a separable differential equation, which can be written

$$\int P^{-2/3}dP = \int 0.3\,e^{-0.01t}dt \qquad \text{or} \qquad 3\,P^{1/3}(t) = -30\,e^{-0.01t} + 3C.$$

It follows that $P^{1/3}(t) = -10\,e^{-0.01t} + C$, so $P(t) = \left(C - 10\,e^{-0.01t}\right)^3$. The initial condition $P(0) = 1000 = (C-10)^3$, which implies $C = 20$. The solution is given by

$$P(t) = \left(20 - 10e^{-0.01\,t}\right)^3.$$

b. This population doubles when $P(t) = \left(20 - 10e^{-0.01\,t}\right)^3 = 2000$, so $20 - 10e^{-0.01\,t} = 10\sqrt[3]{2}$ or $e^{-0.01\,t} = 2 - \sqrt[3]{2}$. It follows that $t = 100\ \ln\left(\frac{1}{2-\sqrt[3]{2}}\right) \approx 30.1$ hr. For large t, $\lim_{t\to\infty} e^{-0.01\,t} = 0$, so $\lim_{t\to\infty} P(t) = 20^3 = 8000$. Thus, there is a horizontal asymptote at $P = 8000$, so the population tends towards this value. The graph of the population can be seen below.



11. a. The change in amount of phosphate, $P(t)$, is found by adding the amount entering and subtracting the amount leaving.

$$\frac{dP}{dt} = 200 \cdot 10 - 200 \cdot c(t),$$

where $c(t)$ is the concentration in the lake with $c(t) = P(t)/10,000$. By dividing the equation by the volume, the concentration equation is given by

$$\frac{dc}{dt} = 0.2 - 0.02c = -0.02(c - 10), \qquad c(0) = 0.$$

With the substitution $z(t) = c(t) - 10$, the equation above reduces to the problem

$$\frac{dz}{dt} = -0.02z, \qquad z(0) = -10,$$

which has the solution $z(t) = -10\,e^{-0.02\,t}$. Thus, the concentration is given by

$$c(t) = 10 - 10\,e^{-0.02\,t}.$$

b. The differential equation describing the growth of the algae is given by

$$\frac{dA}{dt} = 0.5(1 - e^{-0.02\,t})A^{2/3}.$$

By separating variables, we see

$$\begin{aligned} \int A^{-2/3}dA &= 0.5\int (1 - e^{-0.02\,t})dt \\ 3\,A^{1/3}(t) &= 0.5(t + 50\,e^{-0.02\,t}) + C \\ A(t) &= \left(\frac{0.5(t + 50\,e^{-0.02\,t}) + C}{3}\right)^3 \end{aligned}$$

From the initial condition $A(0) = 1000$, we have $1000 = \left(\frac{25+C}{3}\right)^3$. It follows that $C = 5$, so

$$A(t) = \left(\frac{t + 50\,e^{-0.02\,t} + 10}{6}\right)^3.$$

12. a. Write the differential equation $\frac{dw}{dt} = -0.2(w - 80)$, then $z(t) = w(t) - 80$. It follows that

$$\frac{dz}{dt} = -0.2z, \qquad z(0) = -80,$$

with the solution $z(t) = -80e^{-0.2t} = w(t) - 80$. Thus,

$$w(t) = 80\left(1 - e^{-0.2t}\right).$$

For a 40 kg alligator, $w(t) = 40 = 80\left(1 - e^{-0.2t}\right)$ or $40 = 80e^{-0.2t}$, so $e^{0.2t} = 2$ or $0.2t = \ln(2)$. Thus, $t = 5\ln(2) \approx 3.47$ years.

b. The pesticide accumulation is given by

$$\frac{dP}{dt} = 600\left(80\left(1 - e^{-0.2t}\right)\right), \qquad P(0) = 0.$$

The solution is given by

$$P(t) = 48,000\int\left(1 - e^{-0.2t}\right)dt = 48,000\left(t + 5e^{-0.2t}\right) + C.$$

The initial condition gives $P(0) = 0 = 240,000 + C$, so $C = -240,000$. Hence,

$$P(t) = 48,000\left(t + 5e^{-0.2t}\right) - 240,000.$$

The amount of pesticide in the alligator at age 5 is $P(5) = 48,000\left(5+5e^{-1}\right) - 240,000 = 240,000e^{-1} \approx 88291\ \mu$g.

c. The pesticide concentration for a 5 year old alligator is

$$c(5) = \frac{P(5)}{1000w(5)} = \frac{88,291}{80,000\,(1-e^{-1})} \approx 1.75 \text{ ppm}.$$

13. a. The differential equation can be written:

$$\frac{dc}{dt} = -0.004(c-15),$$

so we make the substitution $z(t) = c(t) - 15$. Since $c(0) = 0$, it follows that $z(0) = -15$. The solution of the substituted equation is given by:

$$\begin{aligned} z(t) &= -15e^{-0.004t} = c(t) - 15 \\ c(t) &= 15 - 15e^{-0.004t}. \end{aligned}$$

The limiting concentration satisfies:

$$\lim_{t\to\infty} c(t) = 15 \text{ mg/m}^3.$$

b. We begin by separating variables, which gives:

$$\begin{aligned} \int \frac{dc}{c-15} &= -0.001\int (4-\cos(0.0172t))\,dt \\ \ln(c(t)-15) &= -0.001\left(4t - \frac{\sin(0.0172t)}{0.0172}\right) + C \\ c(t) &= 15 + Ae^{-0.001\left(4t-\frac{\sin(0.0172t)}{0.0172}\right)} \end{aligned}$$

It is easy to see that the initial condition $c(0) = 0$ implies that $A = -15$. Thus, the solution to this problem is given by:

$$c(t) = 15 - 15\,e^{-0.001(4t-58.14\sin(0.0172t))}$$

14. a. We separate variables, so

$$\int M^{-3/4}dM = -k\int dt \quad \text{or} \quad 4M^{1/4} = -kt + 4C$$

$$M(t) = \left(C - \frac{k}{4}t\right)^4$$

From the initial condition, $M(0) = 16 = C^4$, it follows that $C = 2$. From the information that $M(10) = 1 = (2 - 10k/4)^4$, we have $k = 0.4$, so

$$M(t) = (2 - 0.1t)^4.$$

The fruit vanishes in 20 days.

b. We separate variables again to find:

$$\int M^{-3/4} dM = -0.8 \int e^{-0.02t} dt \quad \text{or} \quad 4M^{1/4} = \frac{0.8}{0.02} e^{-0.02t} + 4C$$

$$M(t) = \left(10e^{-0.02t} + C\right)^4.$$

From the initial condition, $M(0) = 16 = (10 + C)^4$, it follows that $C = -8$, so

$$M(t) = \left(10e^{-0.02t} - 8\right)^4.$$

Solving $10e^{-0.02t} = 8$, which is when the fruit vanishes, we find $t = 50 \ln(5/4)$. Thus, the fruit vanishes in 11.157 days.

15. a. The general solution to the Malthusian growth problem with the initial condition $P(0) = 60$ is

$$P(t) = 60\, e^{rt}.$$

We are given that 2 weeks later $P(2) = 80 = 60\, e^{2r}$, so it follows that $r = \frac{1}{2} \ln\left(\frac{4}{3}\right) = 0.14384$. This gives the solution:

$$P(t) = 60\, e^{0.14384\, t}.$$

It is easy to see that the population doubles when $120 = 60\, e^{0.14384\, t}$, so $0.14384\, t_d = \ln(2)$ or the doubling time is

$$t_d = \frac{\ln(2)}{r} = 4.819 \text{ weeks}.$$

b. We begin by separating variables, so the general solution satisfies:

$$\int \frac{dP}{P} = \int (a - b\,t)\, dt \quad \text{or} \quad \ln(P(t)) = a\,t - \frac{bt^2}{2} + C \quad \text{or} \quad P(t) = e^C e^{a\,t - \frac{bt^2}{2}}.$$

Since the initial value is $P(0) = 60$, it follows that $e^C = 60$. Thus,

$$P(t) = 60\, e^{a\,t - \frac{bt^2}{2}}.$$

We now use the data at $t = 2$ and 4 weeks. It follows from the solution above that

$$\begin{aligned} 80 &= 60\, e^{2\,a - 2\,b} \\ 90 &= 60\, e^{4\,a - 8\,b}. \end{aligned}$$

We rearrange the terms and take logarithms of both sides to get

$$\begin{aligned} 2\,a - 2\,b &= \ln\left(\tfrac{4}{3}\right) \\ 4\,a - 8\,b &= \ln\left(\tfrac{3}{2}\right). \end{aligned}$$

We solve these equations simultaneously to obtain

$$2\,b = \ln\left(\tfrac{4}{3}\right) - \tfrac{1}{2} \ln\left(\tfrac{3}{2}\right),$$

so $b = 0.042475$. But $a = b + \frac{1}{2}\ln(4/3)$ or $a = 0.1863$. It follows that the solution is

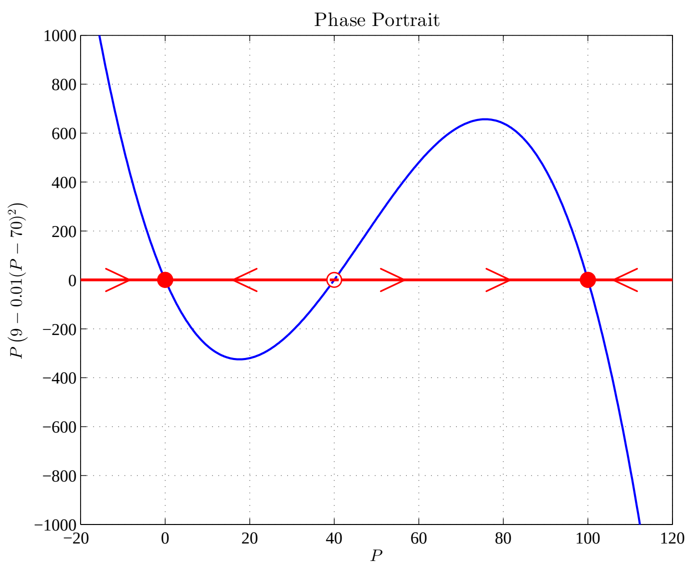
$$P(t) = 60\, e^{0.1863\, t - 0.021237\, t^2}.$$

The population reaches a maximum when the derivative is zero, which occurs when $t_{max} = \frac{a}{b} = 4.3865$, so the maximum population is $P(t_{max}) = 90.286$.

16. (Allee effect) Consider the DE given by the model:

$$\frac{dP}{dt} = P\left(9 - 0.01(P-70)^2\right) = A(P).$$

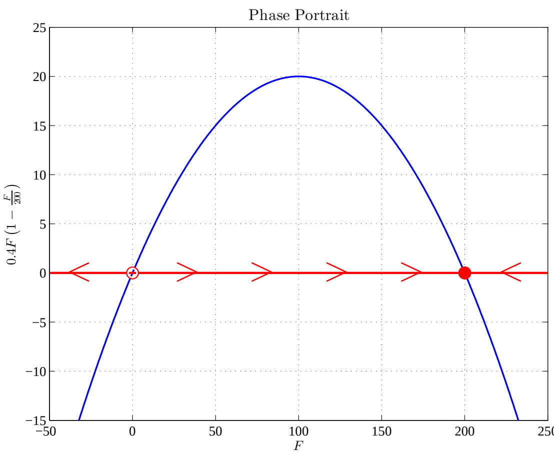
The equilibria of this population model satisfy $P\left(9 - 0.01(P-70)^2\right) = 0$. Thus, $P_e = 0$, 40, and 100. From the phase portrait below, it is easy to see that the equilibria $P_e = 0$ and 100 are stable, while $P_e = 40$ is unstable. The carrying capacity for this population is $P_e = 100$, and the critical threshold number of animals required to avoid extinction is $P_e = 40$.



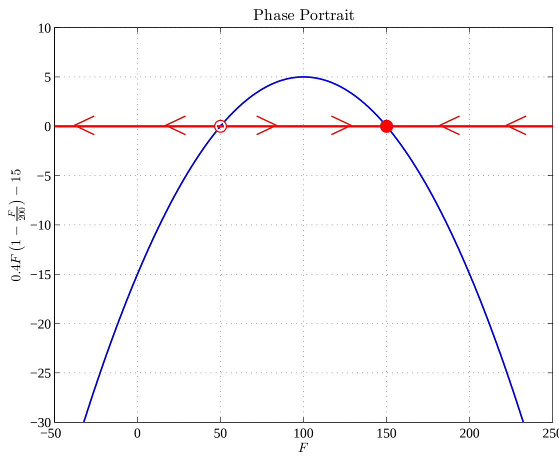
17. a. The solution follows the logistic growth solution seen in 1. d. The solution is

$$F(t) = \frac{10,000}{50 + 150e^{-0.4t}}.$$

b. This is a standard logistic growth model, so the equilibria are $F_e = 0$ and 200 (thousand). Below is a sketch of the function with the phase portrait. The equilibrium $F_e = 0$ is unstable, while the carrying capacity, $F_e = 200$ (thousand), is a stable equilibrium.



Problem 17. b



Problem 17. c

c. With harvesting, the right hand side of the differential equation is written

$$0.4\,F\left(1-\frac{F}{200}\right)-15=-0.002F^2+0.4F-15=-0.002(F-50)(F-150).$$

It follows that the equilibria are $F_e = 50$ and 150 (thousand). Above is a sketch of the function with the phase portrait. The equilibrium $F_e = 50$ (thousand) is the critical number of fish needed to avoid extinction and this equilibrium is unstable. The carrying capacity, $F_e = 150$ (thousand), is a stable equilibrium.

18. a. The ODE for a protein controlled by induction, x, satisfies:

$$\frac{dx}{dt}=\frac{6.5\,x^2}{30+x^2}-0.5\,x=f(x).$$

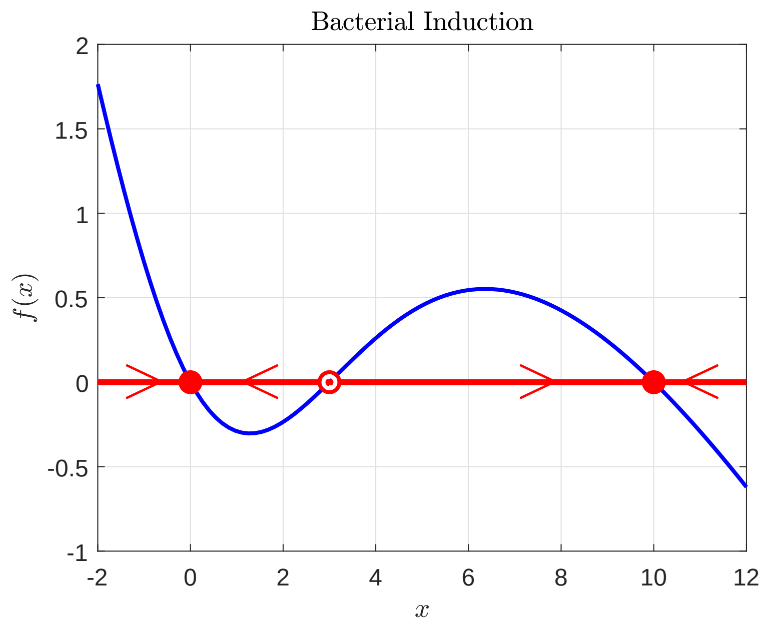
At equilibria,

$$f(x)=\frac{6.5\,x_e^2}{30+x_e^2}-0.5\,x=0, \qquad \text{so} \quad 13\,x_e^2=x_e(30+x_e^2).$$

Rearranging we get:

$$x_e(x_e^2-13x_e+30)=0, \qquad \text{so} \quad x_e(x_e-3)(x_e-10)=0.$$

Since $f(1)=\frac{6.5}{31}-0.5<0$ and the x-intercepts are $x_e = 0,\ 3,\ 10$, we readily draw the graph of the rhs of the ODE with intercepts being equilibria. Below is the graph of the function with the phase portrait on the x-axis, showing all equilibria (with stability): $x_e = 0$ (stable), 3 (unstable), 10 (stable).



b. From this model the concentration of the protein when the gene is *turned on* is $x_e = 10$, and the critical threshold concentration, below which the gene is *turned off* is $x_e = 3$.

Numerical Methods

19. a. The initial value problem:

$$\frac{dy}{dt} = \frac{y}{t} + 2t \qquad \text{or} \qquad \frac{dy}{dt} - \frac{y}{t} = 2t, \qquad y(1) = 5,$$

is a *linear ODE*, which has an integrating factor $\mu(t) = exp\left(-\int \frac{dt}{t}\right) = \frac{1}{t}$. It follows that

$$\frac{d}{dt}\left(\frac{y}{t}\right) = 2, \qquad \text{so integrating} \qquad \frac{y(t)}{t} = 2t + C.$$

With the IC, the solution becomes

$$y(t) = 2t^2 + 3t.$$

b. Euler's method is used to simulate the solution for $t \in [1, 4]$ with stepsizes of $h = 0.2$, 0.1, and 0.05. We compute the percent error between the Euler approximation and the actual solution at $t = 2$, 3, and 4 with **6** significant figures for the Euler approximations and **3** significant figures for the percent errors.

	Actual	Euler	Error	Euler	Error	Euler	Error
t_n		$h = 0.2$	%	$h = 0.1$	%	$h = 0.05$	%
2	14	13.4835	-3.690	13.7325	-1.91	13.8638	-0.973
3	27	25.7581	-4.60	26.3604	-2.37	26.6754	-1.20
4	44	41.8969	-4.78	42.9203	-2.45	43.4529	-1.24

c. Improved Euler's method is used to simulate the solution for $t \in [1, 4]$ with stepsizes of $h = 0.2$, 0.1, and 0.05. We compute the percent error between the Improved Euler approximation and the actual solution at $t = 2$, 3, and 4, using **6** significant figures for the Improved Euler approximations and **3** significant figures for the percent errors.

	Actual	Im Euler	Error	Im Euler	Error	Im Euler	Error
t_n		$h = 0.2$	%	$h = 0.1$	%	$h = 0.05$	%
2	14	13.9655	-0.246	13.9907	-0.0664	13.9976	-0.0171
3	27	26.9299	-0.260	26.9813	-0.0693	26.9952	-0.0178
4	44	43.894	-0.241	43.9718	-0.0641	43.9927	-0.0166

20. The radioactive model given by the IVP:

$$\frac{dR}{dt} = -0.05R + 0.2e^{-0.01t}, \qquad R(0) = 10,$$

is solved both exactly and numerically.

a. The DE is linear and can be written:

$$\frac{dR}{dt} + 0.05R = 0.2e^{-0.01t} \qquad \text{with} \qquad \mu(t) = e^{0.05t}.$$

It follows that it can be written:

$$\frac{d}{dt}\left(e^{0.05t}R(t)\right) = 0.2e^{0.04t} \qquad \text{or} \qquad e^{0.05t}R(t) = 0.2\int e^{0.04t}dt = 5e^{0.04t} + C.$$

Thus, $R(t) = 5e^{-0.01t} + Ce^{-0.05t}$, with $R(0) = 10 = 5 + C$. The solution is:

$$R(t) = 5e^{-0.05t} + 5e^{-0.01t}.$$

b. Euler's method is used to simulate the solution for $t \in [0, 5]$ with stepsizes of $h = 1$, 0.5, and 0.25. We compute the percent error between the Euler approximation and the actual solution at $t = 1$, 2, 3, 4, and 5 with **6** significant figures for the Euler approximations and **3** significant figures for the percent errors.

	Actual	Euler	Error	Euler	Error	Euler	Error
t_n		$h = 1$	%	$h = 0.5$	%	$h = 0.25$	%
1	9.70640	9.7	-0.0659	9.70325	-0.0324	9.70484	-0.0161
2	9.42518	9.41301	-0.129	9.41919	-0.0635	9.42221	-0.0315
3	9.15577	9.13840	-0.190	9.14722	-0.0933	9.15153	-0.0463
4	8.89760	8.87557	-0.248	8.88676	-0.122	8.89222	-0.0605
5	8.65015	8.62395	-0.303	8.63725	-0.149	8.64375	-0.0740

c. Improved Euler's method is used to simulate the solution for $t \in [0.5]$ with stepsizes of $h = 1$, 0.5, and 0.25. We compute the percent error between the Improved Euler approximation and the actual solution at $t = 1$, 2, 3, 4, and 5, using **6** significant figures for the Improved Euler approximations and **3** significant figures for the percent errors.

	Actual	Im Euler	Error	Im Euler	Error	Im Euler	Error
t_n		$h = 1$	%	$h = 0.5$	%	$h = 0.25$	%
1	9.70640	9.70650	0.00112	9.70642	0.000275	9.70640	0.0000681
2	9.42518	9.42539	0.00220	9.42523	0.000539	9.42519	0.000134
3	9.15577	9.15606	0.00323	9.15584	0.000793	9.15579	0.000196
4	8.89760	8.89798	0.00422	8.89769	0.00104	8.89762	0.000257
5	8.65015	8.65060	0.00517	8.65026	0.00127	8.65018	0.000314