HW 6

10

5.5.1. A Sturm-Liouville eigenvalue problem is called self-adjoint if

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0$$

since then $\int_a^b [uL(v) - vL(u)]\ dx = 0$ for any two functions u and v satisfying the boundary conditions. Show that the following yield self-adjoint problems.

(c) $\frac{d\phi}{dx}(0) - h\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(d) $\phi(a) = \phi(b)$ and $p(a)\frac{d\phi}{dx}(a) = p(b)\frac{d\phi}{dx}(b)$

10

5.5.5. Consider

$$L = \frac{d^2}{dx^2} + 6\frac{d}{dx} + 9.$$

(a) Show that $L(e^{rx}) = (r+3)^2 e^{rx}$.

(b) Use part (a) to obtain solutions of $L(y) = 0$ (a second-order constant-coefficient differential equation).

(c) If z depends on x and a parameter r, show that

$$\frac{\partial}{\partial r}L(z) = L\left(\frac{\partial z}{\partial r}\right).$$

(d) Using part (c), evaluate $L(\partial z/\partial r)$ if $z = e^{rx}$.

(e) Obtain a second solution of $L(y) = 0$, using part (d).

15

5.5.8. Consider a fourth-order linear differential operator,

$$L = \frac{d^4}{dx^4}.$$

(a) Show that $uL(v) - vL(u)$ is an exact differential.

(b) Evaluate $\int_0^1 [uL(v) - vL(u)]\ dx$ in terms of the boundary data for any functions u and v.

(c) Show that $\int_0^1 [uL(v) - vL(u)]\ dx = 0$ if u and v are any two functions satisfying the boundary conditions

$$\begin{aligned} \phi(0) &= 0 & \phi(1) &= 0 \\ \frac{d\phi}{dx}(0) &= 0 & \frac{d^2\phi}{dx^2}(1) &= 0. \end{aligned}$$

(d) Give another example of boundary conditions such that

$$\int_0^1 [uL(v) - vL(u)]\ dx = 0.$$

(e) For the eigenvalue problem [using the boundary conditions in part (c)]

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

HW 6 (cont.)

15

5.5.11. *(a) Suppose that

$$L = p(x)\frac{d^2}{dx^2} + r(x)\frac{d}{dx} + q(x).$$

Consider

$$\int_a^b vL(u)\,dx.$$

By repeated integration by parts, determine the adjoint operator L^* such that

$$\int_a^b [uL^*(v) - vL(u)]\,dx = H(x)\Big|_a^b.$$

What is $H(x)$? Under what conditions does $L = L^*$, the **self-adjoint** case? [*Hint:* Show that

$$L^* = p\frac{d^2}{dx^2} + \left(2\frac{dp}{dx} - r\right)\frac{d}{dx} + \left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right)\Bigg].$$

(b) If

$$u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) + u(L) = 0,$$

what boundary conditions should $v(x)$ satisfy for $H(x)|_0^L = 0$, called the adjoint boundary conditions?

5.6.1. Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of

(a) $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\phi(1) = 0$

20

5.8.5. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = -hu(L,t)$, and $u(x,0) = f(x)$.

8 (a) Solve if $h > 0$.

12 (b) Solve if $h < 0$.

25

5.8.8. Consider the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \begin{aligned} \phi(0) - \frac{d\phi}{dx}(0) &= 0 \\ \phi(1) + \frac{d\phi}{dx}(1) &= 0. \end{aligned}$$

5 (a) Using the Rayleigh quotient, show that $\lambda \geq 0$. Why is $\lambda > 0$?

5 (b) Prove that eigenfunctions corresponding to different eigenvalues are orthogonal.

10 *(c) Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues.

HW 6 (cont.)

5 (d) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with

$$\begin{aligned} u(0,t) - \frac{\partial u}{\partial x}(0,t) &= 0 \\ u(1,t) + \frac{\partial u}{\partial x}(1,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

You may call the relevant eigenfunctions $\phi_n(x)$ and assume that they are known.

5 5.8.11. Determine all negative eigenvalues for

$$\frac{d^2\phi}{dx^2} + 5\phi = -\lambda\phi \text{ with } \phi(0) = 0 \text{ and } \phi(\pi) = 0.$$