Spring 2020 Math 531 Take-Home Exam 2 **Due Thur April 23, 2020**

I, ________________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

For all of the problems below, show all of your work and perform all integrations that can be readily done. Use orthogonality to eliminate any zero coefficients. State clearly your reference for any short-cutted solutions to Sturm-Liouville problems.

1. a. Consider a one-dimensional rod that is insulated along its edges with a length of 5 cm. This rod satisfies the PDE:

$$\frac{\partial u}{\partial t} = 0.5\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad 0 < x < 5.$$

This rod exchanges heat via Newton's Law of Cooling with the external environmental temperature being 0°F at the left end and is insulated at the right end. It follows that the boundary conditions are:

$$\frac{\partial u(0,t)}{\partial x} = 0.1u(0,t) \qquad \text{and} \qquad \frac{\partial u(5,t)}{\partial x} = 0.$$

Assume that the initial temperature in the rod satifies:

$$u(x,0) = 50 - 10x, \qquad 0 < x < 5.$$

Find the solution to this problem. For the Sturm-Liouville problem, state clearly your eigenfunction using a single trig function. (Hint: You may want to center your solution around $x = 5$.) Also, give the numerical values of the eigenvalues: λ_1, λ_2, λ_5, λ_{10}, and λ_{20}. Perform the integrations (or let Maple compute them) for the Fourier coefficients.

b. Create a graphic simulation showing the 3D plot of temperature as a function of t and x, using 50 terms in your series to approximate the solution with $t \in [0, 40]$. Be sure to include a copy of your program.

2. Find the steady-state temperature in a cube, which satisfies:

$$\nabla^2 u(x,y,z) = 0, \qquad 0 < x < 2, \quad 0 < y < 2, \quad 0 < z < 2.$$

The cube is insulated on the faces with $x = 0$ and $y = 2$. The cube is kept at 0°C on the faces with $x = 2$ and $z = 0$ and kept at T_0 when $y = 0$. Finally, it satisfies Newton's law of cooling on the other face ($z = 2$) with

$$-k\frac{\partial u(x,y,2)}{\partial z} = hu(x,y,2), \qquad h > 0, \quad k > 0.$$

3. a. A can of beer at room temperature (25°C) is almost submersed in ice water (0°C). Find the steady-state temperature of the beer assuming it satisfies Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0, \qquad 0 < r < 1.5, \quad 0 < z < 5,$$

with the boundary conditions:

$$u(1.5, z) = 0, \quad u(r, 0) = 0, \quad u(r, 5) = 25.$$

You can assume there is infinite ice. For **extra-credit**, assume that you pour this beer into a glass (beer becomes well-mixed, so takes the average steady-state temperature), don't assume any heat transfer from the glass, then use 50 terms in your solution to determine what is the average temperature of the beer.

b. In this part of the problem, we want to know the time evolution of the cooling of the beer. The can of beer satisfies the heat equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right), \qquad 0 < r < 1.5, \quad 0 < z < 5, \quad t > 0,$$

with the boundary conditions:

$$u(1.5, z, t) = 0, \quad u(r, 0, t) = 0, \quad u(r, 5, t) = 25,$$

and initial condition:

$$u(r, z, 0) = 25.$$

Find the temperature of the beer, $u(r, z, t)$ for all $t > 0$. That is, state clearly your Sturm-Liouville problem(s) and any orthogonality relationships. Then solve this problem, showing the full Fourier series solution with the Fourier coefficients from the initial condition.

4. Consider heat conduction in a sphere given by:

$$\frac{\partial u}{\partial t} = \frac{k}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right), \qquad 0 < \rho < a, \quad t > 0,$$

with the boundary and initial conditions:

$$u(a, t) = 0, \quad u(\rho, 0) = T_0.$$

Solve this equation noting any other boundary conditions you might need to apply. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships. (Hint: You might want to try the change of variables given by $u(\rho, t) = v(\rho, t)/\rho$ to create a simpler problem.)

5. Consider the heat equation on a circular region given by:

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \qquad 0 < r < 5, \quad -\pi < \theta < \pi, \quad t > 0.$$

Assume the boundary and initial conditions:

$$u(5,\theta,t) = 0 \qquad \text{and} \qquad u(r,\theta,0) = r^2\big(20 - 10\sin(6\theta)\big).$$

State clearly the implicit boundary conditions. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships. Solve this problem (showing the full Fourier series solution before applying the initial condition), then using orthogonality relative to the initial condition, reduce the Fourier series solution. (**Don't** try to reduce your integrals in r.) Clearly state which Fourier coefficients are **zero**.

6. Consider heat conduction in a sphere given by:

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right)\right), \quad 0 < \rho < a,\ 0 \le \phi \le \pi,\ t > 0,$$

with the boundary and initial conditions:

$$u(a,\phi,t) = 0, \qquad u(\rho,\phi,0) = F(\rho)\big(2 - \cos(\phi)\big).$$

State clearly the implicit boundary conditions. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships. Solve this problem, showing the full Fourier series solution before applying the initial condition. Use the Legendre polynomials with the initial condition to reduce the Fourier series solution. (Hint: Express the initial condition as a sum of Legendre polynomials in $\cos(\phi)$.) Clearly state which Fourier coefficients are **zero**.