23. In this problem we discuss the global truncation error associated with the Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$. Assuming that the functions f and f_y are continuous in a closed, bounded region R of the ty-plane that includes the point (t_0, y_0), it can be shown that there exists a constant L such that $|f(t, y) - f(t, \tilde{y}| < L|y - \tilde{y}|$, where (t, y) and $(t, \tilde{y})$ are any two points in R with the same t coordinate (see Problem 15 of Section 2.8). Further, we assume that f_t is continuous, so the solution ϕ has a continuous second derivative.

(a) Using Eq. (20), show that

$$|E_{n+1}| \le |E_n| + h|f[t_n, \phi(t_n)] - f(t_n, y_n)| + \tfrac{1}{2}h^2|\phi''(\bar{t}_n)| \le \alpha|E_n| + \beta h^2, \tag{i}$$

where $\alpha = 1 + hL$ and $\beta = \max|\phi''(t)|/2$ on $t_0 \le t \le t_n$.

(b) Assume that if $E_0 = 0$, and if $|E_n|$ satisfies Eq. (i), then $|E_n| \le \beta h^2(\alpha^n - 1)/(\alpha - 1)$ for $\alpha \ne 1$. Use this result to show that

$$|E_n| \le \frac{(1 + hL)^n - 1}{L}\beta h. \tag{ii}$$

Equation (ii) gives a bound for $|E_n|$ in terms of h, L, n, and β. Notice that for a fixed h, this error bound increases with increasing n; that is, the error bound increases with distance from the starting point t_0.

(c) Show that $(1 + hL)^n \le e^{nhL}$; hence

$$|E_n| \le \frac{e^{nhL} - 1}{L}\beta h.$$

If we select an ending point T greater than t_0 and then choose the step size h so that n steps are required to traverse the interval $[t_0, T]$, then $nh = T - t_0$, and

$$|E_n| \le \frac{e^{(T-t_0)L} - 1}{L}\beta h = Kh,$$

which is Eq. (25). Note that K depends on the length $T - t_0$ of the interval and on the constants L and β that are determined from the function f.