1. a. The differential equation, $y'' - y = 4t^2 + 6e^{-t}$, is readily solved using the method of undetermined coefficients. The characteristic equation is $\lambda^2 - 1 = 0$, so $\lambda = \pm 1$. Thus, the homogeneous solution is

$$y_c(t) = c_1 e^t + c_2 e^{-t}.$$

It follows that an appropriate guess for the particular solution is

$$y_p(t) = A_2 t^2 + A_1 t + A_0 + Bte^{-t} \quad \text{with} \quad y_p''(t) = 2A_2 + B(t-2)e^{-t}.$$

Thus,

$$y_p'' - y_p = 2A_2 + B(t-2)e^{-t} - (A_2 t^2 + A_1 t + A_0 + Bte^{-t}) = 4t^2 + 6e^{-t}.$$

Matching coefficients gives $A_2 = -4$ from t^2, $A_1 = 0$ from t, $A_0 = 2A_2 = -8$ from t^0, and finally, $-2B = 6$ or $B = -3$. The general solution satisfies:

$$y(t) = c_1 e^t + c_2 e^{-t} - 4t^2 - 8 - 3te^{-t}.$$

b. The differential equation, $y'' - 2y' + 2y = 4\cos(t)$, is readily solved using the method of undetermined coefficients. The characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$, so $\lambda = 1 \pm i$. Thus, the homogeneous solution is

$$y_c(t) = e^t(c_1 \cos(t) + c_2 \sin(t)).$$

It follows that an appropriate guess for the particular solution is

$$y_p(t) = A\cos(t) + B\sin(t) \quad \text{with} \quad y_p'(t) = -A\sin(t) + B\cos(t) \quad \text{and} \quad y_p''(t) = -A\cos(t) - B\sin(t).$$

Thus,

$$\begin{aligned} y_p'' - 2y_p' + 2y_p &= -A\cos(t) - B\sin(t) - 2(-A\sin(t) + B\cos(t)) + 2(A\cos(t) + B\sin(t)) \\ &= 4\cos(t). \end{aligned}$$

Matching coefficients for $\cos(t)$ gives $-A - 2B + 2A = 4$ and for $\sin(t)$ gives $-B + 2A + 2B = 0$. Thus, $B = -2A$, so $5A = 4$ or $A = \frac{4}{5}$ and $B = -\frac{8}{5}$. The general solution satisfies:

$$y(t) = e^t(c_1 \cos(t) + c_2 \sin(t)) + \frac{4}{5}\cos(t) - \frac{8}{5}\sin(t).$$

c. The differential equation, $y'' + 9y = 18\tan(3t)$, requires the variation of parameters. The characteristic equation is $\lambda^2 + 9 = 0$, so $\lambda = \pm 3i$. Thus, the homogeneous solution is

$$y_c(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

The Wronskian satisfies:

$$W[y_1, y_2](t) = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3(\cos^2(3t) + \sin^2(3t)) = 3.$$

The variation of parameters gives

$$\begin{aligned}
y_p(t) &= -\cos(3t)\int^t \frac{18\sin(3s)\tan(3s)}{3}ds + \sin(3t)\int^t \frac{18\cos(3s)\tan(3s)}{3}ds \\
&= -6\cos(3t)\int^t \frac{\sin^2(3s)}{\cos(3t)}ds + 6\sin(3t)\int^t \sin(3s)ds \\
&= -6\cos(3t)\int^t (\sec(3s)-\cos(3t))ds - 2\sin(3t)\cos(3t)ds \\
&= -2\cos(3t)\ln|\sec(3t)+\tan(3t)| + 2\cos(3t)\sin(3t) - 2\sin(3t)\cos(3t)ds \\
&= -2\cos(3t)\ln|\sec(3t)+\tan(3t)|
\end{aligned}$$

The general solution satisfies:

$$y(t) = c_1\cos(3t) + c_2\sin(3t) - 2\cos(3t)\ln|\sec(3t)+\tan(3t)|.$$

d. The differential equation, $y'' + 4y = 8\csc(2t)$, requires the variation of parameters. The characteristic equation is $\lambda^2 + 4 = 0$, so $\lambda = \pm 2i$. Thus, the homogeneous solution is

$$y_c(t) = c_1\cos(2t) + c_2\sin(2t).$$

The Wronskian satisfies:

$$W[y_1, y_2](t) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2(\cos^2(2t) + \sin^2(2t)) = 2.$$

The variation of parameters gives

$$\begin{aligned}
y_p(t) &= -\cos(2t)\int^t \frac{8\sin(2s)\csc(2s)}{2}ds + \sin(2t)\int^t \frac{8\cos(2s)\csc(2s)}{2}ds \\
&= -4\cos(2t)\int^t 1\cdot ds + 4\sin(2t)\int^t \frac{\cos(2s)}{\sin(2s)}ds \\
&= -4t\cos(2t) + 2\sin(2t)\ln|\sin(2t)|
\end{aligned}$$

The general solution satisfies:

$$y(t) = c_1\cos(2t) + c_2\sin(2t) - 4t\cos(2t) + 2\sin(2t)\ln|\sin(2t)|.$$

e. The differential equation,

$$y'' + 4y' + 4y = \frac{e^{-2t}}{t^2},$$

requires the variation of parameters. The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$, so $\lambda = -2$ (repeated root). Thus, the homogeneous solution is

$$y_c(t) = (c_1 + c_2 t)e^{-2t}.$$

The Wronskian satisfies:

$$W[y_1, y_2](t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}.$$

The variation of parameters gives

$$\begin{aligned} y_p(t) &= -e^{-2t}\int^t \frac{se^{-2s}e^{-2s}}{s^2e^{-4s}}ds + te^{-2t}\int^t \frac{e^{-2s}e^{-2s}}{s^2e^{-4s}}ds \\ &= -e^{-2t}\int^t \frac{1}{s}ds + te^{-2t}\int^t \frac{1}{s^2}ds \\ &= -e^{-2t}\ln|t| - e^{-2t} \end{aligned}$$

The general solution satisfies:

$$y(t) = (c_1 + c_2t)e^{-2t} - e^{-2t}(\ln|t| + 1).$$

f. The differential equation, $t^2y'' + 7ty' + 5y = 6t$, requires the variation of parameters. The left hand side is a Cauchy-Euler equation, so attempt solutions of the form $y(t) = t^\lambda$, leaving a characteristic equation $\lambda(\lambda-1) + 7\lambda + 5 = \lambda^2 + 6\lambda + 5 = 0$ or $\lambda_1 = -5$ and $\lambda_2 = -1$. Thus, the homogeneous solution is

$$y_c(t) = c_1t^{-5} + c_2t^{-1}.$$

The Wronskian satisfies:

$$W[y_1, y_2](t) = \begin{vmatrix} t^{-5} & t^{-1} \\ -5t^{-6} & -t^{-2} \end{vmatrix} = 4t^{-7}.$$

Note that the nonhomogeneous term is written $g(t) = \frac{6}{t}$. The variation of parameters gives

$$\begin{aligned} y_p(t) &= -t^{-5}\int^t \frac{6(s^{-1})}{4s(s^{-7})}ds + t^{-1}\int^t \frac{6(s^{-5})}{4s(s^{-7})}ds \\ &= -\frac{3}{2}t^{-5}\int^t s^5ds + \frac{3}{2}t^{-1}\int^t sds \\ &= -\frac{3}{2t^5}\left(\frac{t^6}{6}\right) + \frac{3}{2t}\left(\frac{t^2}{2}\right) = \frac{1}{2}t \end{aligned}$$

The general solution satisfies:

$$y(t) = c_1t^{-5} + c_2t^{-1} + \frac{1}{2}t.$$

2. The differential equation is:

$$(1-t)y'' + ty' - y = 2(t-1)^2e^{-t} \qquad \text{or} \qquad y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = -2(t-1)e^{-t}.$$

Since $y_1(t) = t$, then $y_1'(t) = 1$ and $y_1''(t) = 0$. Substituting into the RHS of the equation above gives:

$$0 + \frac{t}{1-t} - \frac{1}{1-t}t = 0,$$

so satisfies the homogeneous equation. Similarly, if $y_2(t) = e^t$, then $y_2'(t) = e^t$ and $y_2''(t) = e^t$. Substituting into the RHS of the equation above gives:

$$e^t \frac{1-t}{1-t} + \frac{te^t}{1-t} - \frac{e^t}{1-t} = 0,$$

so $y_2(t)$ satisfies the homogeneous problem. Solving the general problem requires finding a particular solution using the variation of parameters. The Wronskian satisfies:

$$W[y_1, y_2](t) = \begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix} = (t-1)e^t.$$

The variation of parameters gives

$$\begin{aligned} y_p(t) &= -t \int^t -\frac{e^s(2(s-1)e^{-s})}{(s-1)e^s} ds + e^t \int^t -\frac{s(2(s-1)e^{-s})}{(s-1)e^s} ds \\ &= t \int^t 2e^{-s} ds - e^t \int^t 2se^{-2s} ds \\ &= -2te^{-t} + \left(t + \frac{1}{2}\right) e^{-t} = \left(\frac{1}{2} - t\right) e^{-t} \end{aligned}$$

The general solution satisfies:

$$y(t) = c_1 t + c_2 e^t + \left(\frac{1}{2} - t\right) e^{-t}.$$

3. a. Consider $y'' - 9y = 2te^{-3t}\sin(t) + 6t^2 e^{3t}$. The solution of the homogeneous equation is

$$y_c(t) = c_1 e^{-3t} + c_2 e^{3t}.$$

The guess that one makes for the method of undetermined coefficients is

$$y_p(t) = e^{-3t}\left((A_1 t + A_0)\cos(t) + (B_1 t + B_0)\sin(t)\right) + t\left(C_2 t^2 + C_1 t + C_0\right) e^{3t}.$$

b. Consider $y'' + 2y' + 2y = 4t^2 e^{-t} + 3te^{-t}\cos(t)$. The solution of the homogeneous equation is

$$y_c(t) = e^{-t}\left(c_1 \cos(t) + c_2 \sin(t)\right).$$

The guess that one makes for the method of undetermined coefficients is

$$y_p(t) = \left(A_2 t^2 + A_1 t + A_0\right) e^{-t} + te^{-t}\left((B_1 t + B_0)\cos(t) + (C_1 t + C_0)\sin(t)\right).$$

4. a. Consider

$$y'' - 4y = 16\, e^{-2t}, \qquad y(0) = 1, \quad y'(0) = 2.$$

Laplace transforms give:

$$s^2Y(s) - s - 2 - 4Y(s) = \frac{16}{s+2} \qquad \text{or} \qquad Y(s) = \frac{s+2}{s^2-4} + \frac{16}{(s+2)(s^2-4)}.$$

Partial fractions on the second expression give

$$\frac{16}{(s+2)(s^2-4)} = \frac{A}{(s+2)^2} + \frac{B}{s+2} + \frac{C}{s-2} \quad \text{or} \quad 16 = A(s-2) + B(s+2)(s-2) + C(s+2)^2.$$

From $s = -2$, $A = -4$. From $s = 2$, $C = 1$. From s^2, $B = -C = -1$. It follows that

$$Y(s) = \frac{-4}{(s+2)^2} - \frac{1}{s+2} + \frac{2}{s-2}.$$

The inverse Laplace transform gives

$$y(t) = 2e^{2t} - e^{-2t} - 4te^{-2t}.$$

b. Consider

$$y'' - y' - 12y = t^2\delta(t-3), \qquad y(0) = 2, \quad y'(0) = -6.$$

Laplace transforms give:

$$s^2Y(s) - 2s + 6 - sY(s) + 2 - 12Y(s) = 9e^{-3s} \qquad \text{or} \qquad Y(s) = \frac{2(s-4)}{(s-4)(s+3)} + \frac{9e^{-3s}}{(s-4)(s+3)}.$$

Partial fractions on the second expression give

$$\frac{9}{(s-4)(s+3)} = \frac{A}{s-4} + \frac{B}{s+3} \quad \text{or} \quad 9 = A(s+3) + B(s-4).$$

From $s = 4$, $A = \frac{9}{7}$. From $s = -3$, $B = -\frac{9}{7}$. It follows that

$$Y(s) = \frac{2}{s+3} + \frac{9}{7}\left(\frac{e^{-3s}}{s-4} - \frac{e^{-3s}}{s+3}\right).$$

The inverse Laplace transform gives

$$y(t) = 2e^{-3t} + \frac{9}{7}u_3(t)\left(e^{4(t-3)} - e^{-3(t-3)}\right).$$

c. Consider

$$y'' + 2y' + 5y = \begin{cases} 5, & 0 \le t < 4 \\ 0, & t \ge 4 \end{cases} \qquad y(0) = 0, \quad y'(0) = 4.$$

Laplace transforms give:

$$s^2Y(s) - 4 + 2sY(s) + 5Y(s) = \frac{5}{s} - \frac{5e^{-4s}}{s} \qquad \text{or} \qquad Y(s) = \frac{4}{(s+1)^2+4} + \frac{5-5e^{-4s}}{s((s+1)^2+4)}.$$

Partial fractions on the second expression give

$$\frac{5}{s((s+1)^2+4)} = \frac{A}{s} + \frac{B(s+1)+2C}{(s+1)^2+4} \quad \text{or} \quad 5 = A((s+1)^2+4) + Bs(s+1) + 2Cs.$$

From $s = 0$, $A = 1$. From s^2, $B = -A = -1$. From $s = -1$, $5 = 4A - 2C$ or $C = -\frac{1}{2}$. It follows that

$$Y(s) = \frac{4}{(s+1)^2+4} + \left(\frac{1}{s} - \frac{s+1}{(s+1)^2+4} - \frac{1}{2}\left(\frac{2}{(s+1)^2+4}\right)\right)(1 - e^{-4s}).$$

The inverse Laplace transform gives

$$y(t) = 1 + e^{-t}\left(\frac{3}{2}\sin(2t) - \cos(2t)\right) - u_4(t)\left(1 - e^{-(t-4)}\left(\cos(2(t-4)) + \frac{1}{2}\sin(2(t-4))\right)\right).$$

d. Consider

$$y'' + 4y = \cos(t), \qquad y(0) = 1, \quad y'(0) = 0.$$

Laplace transforms give:

$$s^2Y(s) - s + 4Y(s) = \frac{s}{s^2+1} \qquad \text{or} \qquad Y(s) = \frac{s}{s^2+4} + \frac{s}{(s^2+1)(s^2+4)}.$$

Partial fractions on the second expression give

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+2D}{s^2+4} \quad \text{or} \quad s = (As+B)(s^2+4) + (Cs+2D)(s^2+1).$$

From $s = i$, $i = 3(iA+B)$, so $A = \frac{1}{3}$ and $B = 0$. From $s = 2i$, $2i = -3(2iC + 2D)$, so $C = -\frac{1}{3}$ and $D = 0$. It follows that

$$Y(s) = \frac{s}{s^2+4} + \frac{1}{3}\left(\frac{s}{s^2+1}\right) - \frac{1}{3}\left(\frac{s}{s^2+4}\right) = \frac{2}{3}\left(\frac{s}{s^2+4}\right) + \frac{1}{3}\left(\frac{s}{s^2+1}\right).$$

The inverse Laplace transform gives

$$y(t) = \frac{2}{3}\cos(2t) + \frac{1}{3}\cos(t).$$

e. Consider

$$y'' + 2y' - 3y = \begin{cases} t^2, & 0 \le t < 3 \\ 0, & t \ge 3 \end{cases} \qquad y(0) = 0, \quad y'(0) = 0.$$

The function on the RHS can be written $f(t) = t^2 - t^2u_3(t) = t^2 - (t-3)^2u_3(t) - 6(t-3)u_3(t) - 9u_3(t)$. Laplace transforms give:

$$s^2Y(s) + 2Y(s) - 3Y(s) = \frac{2}{s^3} - \frac{2e^{-3s}}{s^3} - 6\frac{e^{-3s}}{s^2} - 9\frac{e^{-3s}}{s}.$$

(Alternately, one could evaluate

$$F(s) = \int_0^3 t^2 e^{-st} dt,$$

which gives the same answer.) It follows that

$$Y(s) = \frac{2 - 2e^{-3s} - 6se^{-3s} - 9s^2e^{-3s}}{s^3(s+3)(s-1)}$$

The partial fractions is tedious but straight forward, giving

$$\begin{aligned}
\frac{2}{s^3(s+3)(s-1)} &= \frac{1}{54(s+3)} + \frac{1}{2(s-1)} - \frac{2}{3s^3} - \frac{4}{9s^2} - \frac{14}{27s} \\
\frac{-2-6s-9s^2}{s^3(s+3)(s-1)} &= -\frac{65}{108(s+3)} - \frac{17}{4(s-1)} + \frac{2}{3s^3} + \frac{22}{9s^2} + \frac{131}{27s}
\end{aligned}$$

The inverse Laplace transform gives the solution

$$\begin{aligned}
y(t) &= \frac{1}{54}e^{-3t} + \frac{1}{2}e^{t} - \frac{1}{3}t^2 - \frac{4}{9}t - \frac{14}{27} \\
&\quad -u_3(t)\left(\frac{65}{108}e^{-3(t-3)} + \frac{17}{4}e^{(t-3)} - \frac{1}{3}(t-3)^2 - \frac{22}{9}(t-3) - \frac{131}{27}\right)
\end{aligned}$$

5. a. If $\mathcal{L}\{f\} = F(s)$, then

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Leibnitz's rule of differentiation of the integral states that

$$\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b \frac{\partial f(x,y)}{\partial x}dy.$$

In other words, the differentiation operator can commute with the integral. Thus,

$$\frac{dF(s)}{ds} = \int_0^\infty \frac{\partial f(t)e^{-st}}{\partial s}dt = -\int_0^\infty t\,f(t)e^{-st}dt.$$

It follows that

$$\mathcal{L}\{t\,f(t)\} = -\frac{dF(s)}{ds}.$$

b. Since $\mathcal{L}\sin(2t) = \frac{2}{s^2+4}$, then

$$-\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2},$$

which is $\mathcal{L}\{t\sin(2t)\}$.

c. For the initial value problem

$$y'' + 4y = 2\cos(2t), \qquad y(0) = -1, \quad y'(0) = 4,$$

we take Laplace transforms and obtain:

$$s^2Y(s) + s - 4 = 4Y(s) = \frac{2s}{s^2+4} \qquad \text{or} \qquad Y(s) = \frac{4-s}{s^2+4} + \frac{2s}{(s^2+4)^2}$$

The inverse Laplace transform gives

$$y(t) = 2\sin(2t) - \cos(2t) + \frac{1}{2}t\sin(2t).$$

6. a. The differential equation can be written

$$y'' + y = \delta(t) + \delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi) + ...$$

The Laplace transform becomes

$$\begin{aligned}\mathcal{L}\{y''\} + \mathcal{L}\{y'\} &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y'\} \\ &= \mathcal{L}\{\delta(t) + \delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi) + ...\}\end{aligned}$$

or

$$(s^2+1)\mathcal{L}\{y\} = 1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + ...$$

Thus,

$$\mathcal{L}\{y\} = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} + \frac{e^{-3\pi s}}{s^2+1} + ...$$

or

$$\begin{aligned}y(t) &= \sin(t) + u_\pi(t)\sin(t-\pi) + u_{2\pi}(t)\sin(t-2\pi) + u_{3\pi}(t)\sin(t-3\pi) + ... \\ &= \sin(t)(1 - u_\pi(t) + u_{2\pi}(t) - u_{3\pi}(t) + ...) = \sin(t)\sum_{n=0}^{\infty}(-1)^n u_{n\pi}(t).\end{aligned}$$

b. The differential equation can be written

$$y'' + y = \delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + ...$$

The Laplace transform becomes

$$\begin{aligned}\mathcal{L}\{y''\} + \mathcal{L}\{y'\} &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y'\} \\ &= \mathcal{L}\{\delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + ...\}\end{aligned}$$

or

$$(s^2+1)\mathcal{L}\{y\} = 1 + e^{-2\pi s} + e^{-4\pi s} + e^{-6\pi s} + ...$$

Thus,

$$\mathcal{L}\{y\} = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} + \frac{e^{-4\pi s}}{s^2+1} + \frac{e^{-6\pi s}}{s^2+1} + ...$$

or

$$\begin{aligned}y(t) &= \sin(t) + u_{2\pi}(t)\sin(t-2\pi) + u_{4\pi}(t)\sin(t-4\pi) + u_{6\pi}(t)\sin(t-6\pi) + ... \\ &= \sin(t)(1 + u_{2\pi}(t) + u_{4\pi}(t) + u_{6\pi}(t) + ...) = \sin(t)\sum_{n=0}^{\infty} u_{2n\pi}(t).\end{aligned}$$

c. In the first case, sine functions alternately cancel each other out. Thus, on alternate intervals of π, the function is positive part of the sine curve followed by an interval of the zero function. The second case is the resonance case. In this case, each interval of 2π has the solution increasing the amplitude of the sine function by 1. This solution becomes unbounded, so the bridge would collapse.