Homework – Taylor's polynomials – Solutions Due Wed. 1/24

1. (**Each part is worth 3 pts**) Find the Taylor Series of

   (a) $f(x) = 1/x$ around $x_0 = 1$.

$$\begin{array}{rclrcl} f^0(x) & = & x^{-1} & f^0(1) & = & 1 \\ f^1(x) & = & (-1)x^{-2} & f^1(1) & = & -1 \\ f^2(x) & = & (-1)(-2)x^{-3} & f^2(1) & = & 2 \\ f^n(x) & = & \frac{(-1)^n(n!)}{x^{n+1}} & f^n(1) & = & (-1)^n(n!) \end{array}$$

$$T(x) = 1 + (-1)(x-1) + \frac{1}{2!}(2)(x-1)^2 + ... + \frac{1}{n!}(-1)^n(n!)(x-1)^n + ...$$

or

$$T(x) = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

   (b) $f(x) = \sqrt{x}$ around $x_0 = 4$.

$$\begin{array}{rclrcl} f^0(x) & = & x^{\frac{1}{2}} & f^0(4) & = & 2 \\ f^1(x) & = & \frac{1}{2}x^{-\frac{1}{2}} & f^1(4) & = & \frac{1}{4} \\ f^2(x) & = & (-\frac{1}{2})(\frac{1}{2})x^{-\frac{3}{2}} & f^2(4) & = & -\frac{1}{2^5} \\ f^3(x) & = & (-\frac{3}{2})(-\frac{1}{2})(\frac{1}{2})x^{-\frac{5}{2}} & f^3(4) & = & \frac{3}{2^8} \\ f^4(x) & = & (-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})(\frac{1}{2})x^{-\frac{7}{2}} & f^4(4) & = & \frac{15}{2^{10}} \end{array}$$

$$T(x) = 2 + \left(\frac{1}{4}\right)(x-4) - \frac{1}{2!}\left(\frac{1}{2^5}\right)(x-4)^2 + \frac{1}{3!}\left(\frac{3}{2^8}\right)(x-4)^3 + ...$$

or

$$T(x) = 2 + \left(\frac{1}{4}\right)(x-4) - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} - \frac{5(x-4)^4}{16384} + ...$$

or

$$T(x) = 2 + \left(\frac{1}{4}\right)(x-4) + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n!2^{3n-1})}\left(\prod_{k=0}^{n-2}(2k+1)\right)(x-4)^n.$$

2. (**Each part is worth 3 pts**) Find values x_0, δ, and M such that

$$\max_{x\in[x_0-\delta,x_0+\delta]}|f(x)| \leq M$$

for

(a) $f(x) = \sin(x)/x^3$ for $x \in [1, 3]$.

The midpoint of the interval is $x_0 = 2$, and the distance from the midpoint to either end is 1, so $\delta = 1$. We know that for any x, $|\sin(x)| \leq 1$. Also, $\frac{1}{x^3}$ is monotonically decreasing for $x \in [1, 3]$, so $\frac{1}{x^3} \leq 1$. It follows that

$$\max_{x\in[1,3]} \left| \frac{\sin(x)}{x^3} \right| \leq \max_{x\in[1,3]} \frac{1}{|x|^3} \leq 1,$$

so $M = 1$. In fact, $\sin(x)/x^3$ is monotonically decreasing, so the best bound is $\sin(1)/1 \approx 0.8415$.

(b) $f(x) = \sqrt{\sin^2(x) + 8}$ for $x \in [2, 6]$.

The midpoint of the interval is $x_0 = 4$, and the distance from the midpoint to either end is 2, so $\delta = 2$. We know that for any x, $\sin^2(x) \leq 1$. It follows that

$$\max_{x\in[2,6]} \left| \sqrt{\sin^2(x) + 8} \right| \leq \left| \sqrt{1 + 8} \right| = 3,$$

so $M = 3$. In fact, $\sin^2(x)$ achieves its max at $x = \frac{3\pi}{2}$, so this function reaches 3 for that x.

3. (**3 pts**) Use the Maclaurin series for e^x, $\cos(x)$, and $\sin(x)$ to demonstrate Euler's formula:

$$e^{ix} = \cos(x) + i \sin(x).$$

The Maclaurin series for e^{ix} is

$$\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty} \frac{1}{(n!)}(ix)^n \\
&= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + ... \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{((2n)!)} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{((2n+1)!)} x^{2n+1} \\
&= \cos(x) + i \sin(x)
\end{aligned}$$

4. (**Each part is worth 3 pts**) We cannot exactly find the integral:

$$\int_0^1 e^{-x^2} dx,$$

but we can approximate it. To do this,

(a) Using a third order Maclaurin series approximation to find an approximation to the integral.

The cubic expansion for $e^{-x^2} = 1 - x^2 + \mathcal{O}(x^4)$, so

$$\int_0^1 e^{-x^2} dx \approx \int_0^1 \left(1 - x^2\right) dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} = 0.66666667$$

(b) We know that:

$$\int_0^1 e^{-x^2}\,dx = \int_0^{1/4} e^{-x^2}\,dx + \int_{1/4}^{1/2} e^{-x^2}\,dx + \int_{1/2}^{3/4} e^{-x^2}\,dx + \int_{3/4}^{1} e^{-x^2}\,dx,$$

To obtain the 4 Taylor series, we need the derivatives of $f(x) = e^{-x^2}$, which are:

$$\begin{aligned} f'(x) &= -2xe^{-x^2} \\ f''(x) &= (4x^2-2)e^{-x^2} \\ f'''(x) &= (12x-8x^3)e^{-x^2} \end{aligned}$$

$$\int_0^{1/4} e^{-x^2}\,dx \approx \int_0^{1/4} \left(1-x^2\right)dx = \left[x - \frac{x^3}{3}\right]_0^{1/4} = \frac{47}{192} \approx 0.2447917$$

Choosing the Taylor series around $x_0 = \frac{1}{4}$

$$\begin{aligned} \int_{1/4}^{1/2} e^{-x^2}\,dx &\approx \int_{1/4}^{1/2} \left(e^{-1/16} - \frac{e^{-1/16}}{2}(x-\frac{1}{4}) - \frac{7e^{-1/16}}{8}(x-\frac{1}{4})^2 + \frac{23e^{-1/16}}{48}(x-\frac{1}{4})^3\right)dx \\ &\approx 0.2163333 \end{aligned}$$

$$\begin{aligned} \int_{1/2}^{3/4} e^{-x^2}\,dx &\approx \int_{1/2}^{3/4} \left(e^{-1/4} - e^{-1/4}(x-\frac{1}{2}) - \frac{e^{-1/4}}{2}(x-\frac{1}{2})^2 + \frac{5e^{-1/4}}{6}(x-\frac{1}{2})^3\right)dx \\ &\approx 0.1689683 \end{aligned}$$

$$\begin{aligned} \int_{3/4}^{1} e^{-x^2}\,dx &\approx \int_{3/4}^{1} \left(e^{-9/16} - \frac{3e^{-9/16}}{2}(x-\frac{3}{4}) + \frac{e^{-9/16}}{8}(x-\frac{3}{4})^2 + \frac{15e^{-9/16}}{16}(x-\frac{3}{4})^3\right)dx \\ &\approx 0.1166297 \end{aligned}$$

The sum of the 4 integrals is approximately 0.7467230.

(c) Matlab says that

$$\int_0^1 e^{-x^2}\,dx = 0.746824132812427,$$

The second answer is much better because the approximations fit the curve more closely than trying to fit the function with only a single quadratic.

5. (**Each part is worth 1 pt**) In a Matlab command window, type in the following:

```
X = ['cat'];
Y = ['food'];
```

Write the output from the following commands

(a) `[X Y]`

(b) `[X'; Y']`

(c) `[X' Y']` (and yes, you should get the program yelling at you, why?)

(d) `Y(1:2:4)`

(e) `Y(1:2:end)`

(f) `X(end-1)`

(g) `[Y 'is good']`

(h) `[X [Y 'is good']]`

(i) `length(Y)`

(j) `length(X)`

Answers are respectively:

(a) catfood

(b) c
a
t
f
o
o
d

(c) Error using Horzcat

Dimensions of matrices being concatenated are not consistent. You have transposed the two matrices and they no longer have the same number of rows.

(d) fo

(e) fo

(f) a

(g) food is good

(h) catfood is good

(i) 4

(j) 3