April 25, 2018 Math 337 Name Key

Note: For full credit you must show intermediate steps in your calculations.

1. (37pts) For the following 2^{nd} order nonhomogeneous differential equations, find the general solution. You may use any technique that works for the problem.

a. $y'' - 3y' = 12t$ c.e. homogen. $\lambda^2 - 3\lambda = \lambda(\lambda-3) = 0$

4 $y_h(t) = c_1 + c_2 e^{3t}$

$\therefore y(t) = c_1 + c_2 e^{3t} - 2t^2 - \frac{4}{3}t$

$y_p(t) = t(At + B)$

7 $y_p'(t) = 2At + B$, $y_p'' = 2A$

Substitute

$2A - 3(2At + B) = 12t$

t: $-6A = 12 \Rightarrow A = -2$

t^0: $2A - 3B = 0 \Rightarrow B = -4/3$

b. $y'' + 25y = 30\sec(5t)$ c.e. $\lambda^2 + 25 = 0$, $\lambda = \pm 5i$

4 $y_h(t) = c_1\cos(5t) + c_2\sin(5t)$

$W[y_1, y_2] = \begin{vmatrix} \cos(5t) & \sin(5t) \\ -5\sin(5t) & 5\cos(5t) \end{vmatrix}$ 2

$= 5(\cos^2(5t) + \sin^2(5t)) = 5$

Var. Param.

$y_p(t) = -\cos(5t)\int^t \frac{\sin(5s)\cdot 30\sec(5s)}{5}ds + \sin(5t)\int^t \frac{\cos(5s)\cdot 30\sec(5s)}{5}ds$

7 $= +\frac{6\cos(5t)}{5}\int \frac{-5\sin(5s)}{\cos(5s)}ds + 6\sin(5t)\int^t ds = \frac{6}{5}\ln|\cos(5t)|\cos(5t) + 6t\sin(5t)$

$y(t) = c_1\cos(5t) + c_2\sin(5t) + \frac{6}{5}\cos(5t)\ln|\cos(5t)| + 6t\sin(5t)$

c. $t^2y'' - ty' - 3y = 8t^3$ Rewrite $y'' - \frac{y'}{t} - \frac{3}{t^2}y = 8t$

Cauchy-Euler $r(r-1) - r - 3 = r^2 - 2r - 3 = (r-3)(r+1) = 0$

4 $y_h(t) = c_1 t^{-1} + c_2 t^3$

$W[y_1, y_2] = \begin{vmatrix} t^{-1} & t^3 \\ -t^{-2} & 3t^2 \end{vmatrix} = 3t + t = 4t$ 2

Var. Param

$y_p(t) = -t^{-1}\int^t \frac{s^3\cdot 8s}{4s}ds + t^3\int^t \frac{s^{-1}\cdot 8s}{4s}ds = -t^{-1}\cdot 2\int s^3 ds + t^3\int^t \frac{2}{s}ds$

7 $= -t^{-1}\left(\frac{2t^4}{4}\right) + 2t^3\ln|t| = -\frac{t^3}{2} + 2t^3\ln|t|$

$y(t) = c_1 t^{-1} + \left(c_2 - \frac{1}{2}\right)t^3 + 2t^3\ln|t|$

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2. (37pts) Solve the following initial value problems using the method of Laplace transforms. (Thus, you must show the problem in the transform space and the solution in t.)

a. $y'' - 4y' + 5y = 50t, \quad y(0) = 1, \quad y'(0) = 0$ Let $\mathcal{L}\{y(t)\} = Y(s)$

$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 5Y(s) = \frac{50}{s^2}$

3

$(s^2 - 4s + 5)Y(s) = s - 4 + \frac{50}{s^2}$

$Y(s) = \frac{s-2}{(s-2)^2+1} - \frac{2}{(s-2)^2+1} + \frac{50}{s^2(s^2-4s+5)}$

$\frac{50}{s^2(s^2-4s+5)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C(s-2)+D}{(s-2)^2+1}$

$s=0: \; 50 = 5A \Rightarrow A = 10$
$s^3: \; B + C = 0$
$s^2: \; A - 4B - 2C + D = 0$
$s^1: \; -4A + 5B = 0 \Rightarrow B = 8$
$s^3 \Rightarrow C = -8$
$s^2 \Rightarrow D = -10 + 32 - 16 = 6$

$50 = A(s^2-4s+5) + Bs(s^2-4s+5) + Cs^2(s-2) + Ds^2$

10 $Y(s) = \frac{s-2}{(s-2)^2+1} - \frac{2}{(s-2)^2+1} - \frac{8(s-2)}{(s-2)^2+1} + \frac{6}{(s-2)^2+1} + \frac{10}{s^2} + \frac{8}{s}$

5 $y(t) = \mathcal{L}^{-1}\{Y(s)\} = -7e^{2t}\cos(t) + 4e^{2t}\sin(t) + 10t + 8$

b. $y'' + 2y' = \begin{cases} 0, & 0 \le t < 5 \\ 4, & t \ge 5 \end{cases} \quad y(0) = 1, \quad y'(0) = 2$ Let $\mathcal{L}\{y(t)\} = Y(s)$

$= 4u_5(t)$

$s^2Y(s) - s - 2 + 2sY(s) - 2 = \frac{4e^{-5s}}{s}$

4

$(s^2 + 2s)Y(s) = s + 4 + \frac{4e^{-5s}}{s}$

$\frac{s+4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$

$s + 4 = A(s+2) + Bs$

$s=0: \; 4 = 2A \Rightarrow A = 2$
$s=-2: \; 2 = -2B \Rightarrow B = -1$

$Y(s) = \frac{s+4}{s(s+2)} + \frac{4e^{-5s}}{s^2(s+2)}$

10

$Y(s) = \frac{2}{s} - \frac{1}{s+2} + e^{-5s}\left(\frac{2}{s^2} - \frac{1}{s} + \frac{1}{s+2}\right)$

$\frac{4}{s^2(s+2)} = \frac{C}{s^2} + \frac{D}{s} + \frac{E}{s+2}$

$4 = C(s+2) + Ds(s+2) + Es^2$

$s=0: \; 4 = 2C \Rightarrow C = 2$ $\quad s^2: \; 0 = D + E$
$s=-2: \; 4 = 4E \Rightarrow E = 1$ $\quad \Rightarrow D = -1$

5 $y(t) = 2 - e^{-2t} + u_5(t)\left(2(t-5) - 1 + e^{-2(t-5)}\right)$

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3. (13pts) For the following nonhomogeneous differential equation, find the homogeneous solution and give the form of the particular solution that you would guess in using the **method of undetermined coefficients**. (**DO NOT** solve this differential equation.)

$$y'' + 4y' + 4y = 4te^{-2t}\sin(2t) - 8t^2e^{-2t}$$

c.e.: $\lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0$

4 $y_h(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

$y_p(t) = e^{-2t}\left[(A_1 t + A_0)\cos(2t) + (B_1 t + B_0 \sin(2t))\right] + t^2\left[c_2 t^2 + c_1 t + c_0\right]e^{-2t}$

4 5

4. (13pts) a. Use the definition of the Laplace transform to find

$$\mathcal{L}\{\sinh(\omega t)\}, \qquad s > \omega.$$

That is, set up and evaluate the appropriate indefinite integrals. Write your answer with one common denominator. Recall that

$$\sinh(\omega t) = \frac{e^{\omega t} - e^{-\omega t}}{2}.$$

$\mathcal{L}\{\sinh(\omega t)\} = \int_0^\infty e^{-st}\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right) dt = \frac{1}{2}\int_0^\infty e^{-t(s-\omega)} dt - \frac{1}{2}\int_0^\infty e^{-t(s+\omega)} dt$

$= \frac{1}{2}\lim_{A\to\infty}\left[\int_0^A e^{-t(s-\omega)} dt - \int_0^A e^{-t(s+\omega)} dt\right]$ $\qquad s > \omega$

6 $= \frac{1}{2}\lim_{A\to\infty}\left[\frac{e^{-t(s-\omega)}}{-(s-\omega)} - \frac{e^{-t(s+\omega)}}{-(s+\omega)}\right]_0^A = \frac{1}{2}\lim_{A\to\infty}\left[\frac{\overset{0}{\cancel{e^{-A(s-\omega)}}}}{-(s-\omega)} - \frac{\overset{0}{\cancel{e^{-A(s+\omega)}}}}{-(s+\omega)}\right]$

$-\frac{1}{2}\left[\frac{1}{-(s-\omega)} - \frac{1}{-(s+\omega)}\right] = \frac{1}{2}\,\frac{2\omega}{s^2-\omega^2} = \frac{\omega}{s^2-\omega^2}$

b. **Use** the result in **Part a** to solve the initial value problem

$$y'' - 16y = 0, \qquad y(0) = 0, \quad y'(0) = 8.$$

$\mathcal{L}\{y(t)\} = Y(s)$

$s^2 Y(s) - 8 - 16Y(s) = 0$

4 $(s^2 - 16)Y(s) = 8 \implies Y(s) = \frac{8}{s^2-16} = 2\,\frac{4}{s^2-16}$

3 $y(t) = 2\sinh(4t)$

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