April 4, 2018 Math 337 Name Key

Note: For full credit you must show intermediate steps in your calculations.

1. (30pts) For the following 1^{st} order systems of differential equations, find the general solution and determine the specific solution to the initial value problem. Sketch the phase portrait for typical solutions, including the specific solution to the initial value problem. When the eigenvectors are real, show the eigenvectors in the phase portrait. State the **type of node**.

a. $\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$x(t) = 3\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + 2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}$

2 $\begin{vmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2) = 0$

$\lambda_1 = -2, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \quad \lambda_2 = 3, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

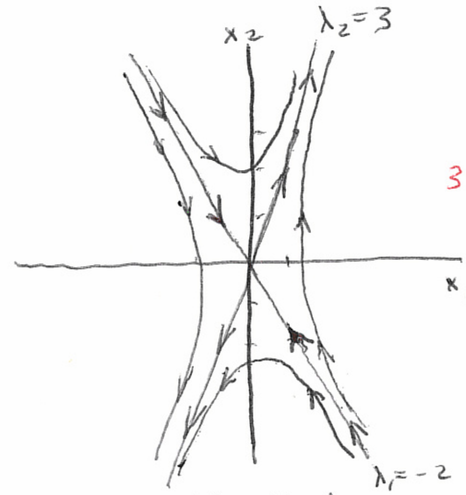
6 $\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \vec{0} \qquad \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \vec{0}$

$x(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}$

3 $x(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 3 \end{pmatrix} c_2 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

$2(c_1 + c_2 = 5)$
$-2c_1 + 3c_2 = 0$
$5c_2 = 10$

$c_2 = 2$
$c_1 = 3$



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Saddle Node 1

b. $\dot{\mathbf{x}} = \begin{pmatrix} -2 & -1 \\ 5 & -4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

2 $\begin{vmatrix} -2-\lambda & -1 \\ 5 & -4-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 + 5 = (\lambda+3)^2 + 4 = 0$

$\lambda = -3 \pm 2i$

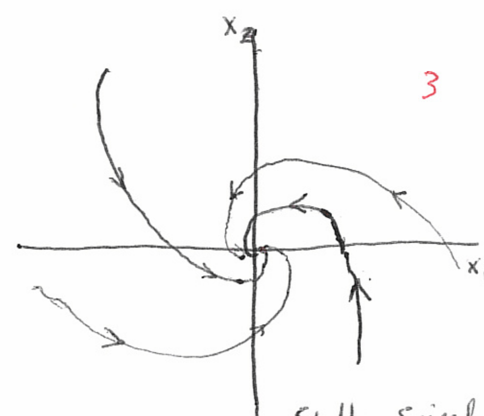
$\lambda_1 = -3 + 2i \qquad v_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$

$\begin{pmatrix} 1-2i & -1 \\ 5 & -1-2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \vec{0}$

6 $(1-2i)\xi_1 - \xi_2 = 0$

$x_1(t) = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{-3t}(\cos(2t) + i\sin(2t))$

$x(t) = e^{-3t}\left[c_1 \begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}\right]$



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Stable Spiral 1

$\begin{pmatrix} -2 & -1 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

3 $\begin{pmatrix} 1 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} 0 \\ -2 \end{pmatrix} c_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$c_1 = 2, \quad c_2 = -1$

$x(t) = e^{-3t}\left[2\begin{pmatrix} \cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix} - \begin{pmatrix} \sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}\right]$

30

2. (25pts) a. Consider the 1^{st} order system of differential equations given by:

$$\dot{\mathbf{x}} = \begin{pmatrix} -2 & \alpha - 1 \\ 1 & -2 \end{pmatrix} \mathbf{x},$$

where α is a parameter. Find the characteristic equation and eigenvalues in terms of α.

$$\begin{vmatrix} -2-\lambda & \alpha-1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 - (\alpha-1) = 0$$

3 $\quad \lambda = -2 \pm \sqrt{\alpha - 1}$

b. There are **two** critical values of α ($\alpha_1 < \alpha_2$), where the qualitative nature of the phase portrait changes. (For example, unstable node to unstable spiral or saddle node to stable node.) Determine values of α where the type of node changes for the origin. Characterize the values of the eigenvalues for $\alpha < \alpha_1$, $\alpha = \alpha_1$, $\alpha \in (\alpha_1, \alpha_2)$, $\alpha = \alpha_2$, and $\alpha > \alpha_2$. State clearly the type of behavior (such as **STABLE NODE**) for each of these values of α.

Critical Values $\alpha = 1$ (Divides complex), $\alpha = 5$ ($\lambda = 0$ e.v.)

$\alpha < 1$: $\lambda = -2 \pm i\omega$ - Stable Spiral

$\alpha = 1$: $\lambda = -2$ - Improper Stable Node

10 $\quad 1 < \alpha < 5$: $\lambda_1 < \lambda_2 < 0$ - Stable Node

$\alpha = 5$: $\lambda_1 = -4$, $\lambda_2 = 0$ - Line of Equil., Stable Equil.

$\alpha > 5$: $\lambda_1 < 0 < \lambda_2$ - Saddle Node

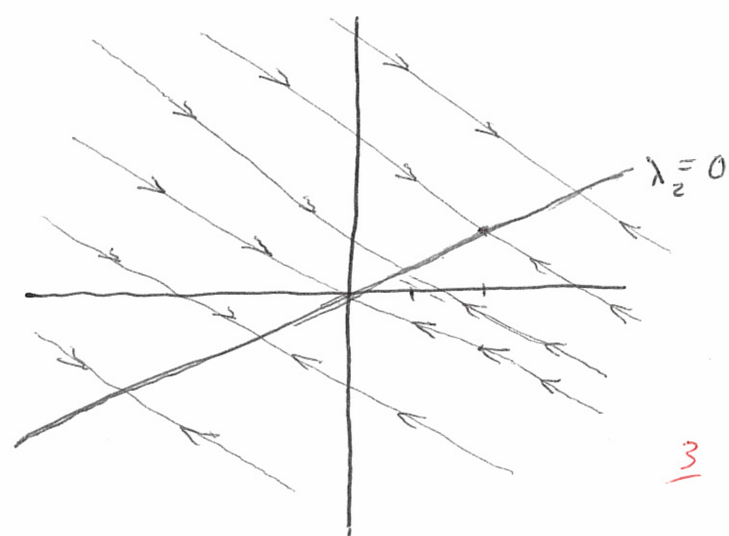
c. For $\alpha = 5$, find the general solution to this equation and sketch a phase portrait.

$\lambda_1 = -4$, $\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \vec{0}$

6 $\quad \lambda_2 = 0$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \vec{0}$



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3 $\quad x(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-4t}$

25

4. (25pts) a. Consider a two species competition model with the species $x(t)$ and $y(t)$. Assume that the model satisfies the system of differential equations given by:

$$\frac{dx}{dt} = 0.1x - 0.001x^2 - 0.002xy,$$
$$\frac{dy}{dt} = 0.2y - 0.0025y^2 - 0.005xy.$$

Find all equilibria.

$x_e(100 - x_e - 2y_e) = 0$ $\quad$ $x_e = 0$ or $x_e + 2y_e = 100$ $\quad$ $-7.5y_e = -300$

$y_e(200 - 2.5y_e - 5x_e) = 0$ $\quad$ $y_e = 0$ or $5x_e + 2.5y_e = 200$ $\quad$ $y_e = 40$

$-5x_e - 10y_e = -500$ $\quad$ $x_e = 20$

7 $\quad (x_e, y_e) = (0,0),\ (0,80),\ (100,0),\ (20,40)$

b. Find the Jacobian matrix, $J(x,y)$, for this system of differential equations. Show the linearized model for populations near the equilibrium where species y is extinct (0) and species $x > 0$. Find the eigenvalues and eigenvectors at this equilibria and determine the type of node.

$$J(x,y) = \begin{pmatrix} 0.1 - 0.002x - 0.002y & -0.002x \\ -0.005y & 0.2 - 0.005y - 0.005x \end{pmatrix}$$

12 $\quad J(100,0) = \begin{pmatrix} -0.1 & -0.2 \\ 0 & -0.3 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 0.2 & -0.2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\lambda_1 = -0.1,\ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\quad \lambda_2 = -0.3,\ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad$ Stable Node

c. Create a diagram in phase space, showing clearly the equilibria and **all** nullclines ($\dot{x}_1 = 0$ or $\dot{x}_2 = 0$). Include at least one typical solution starting from low populations of both species.

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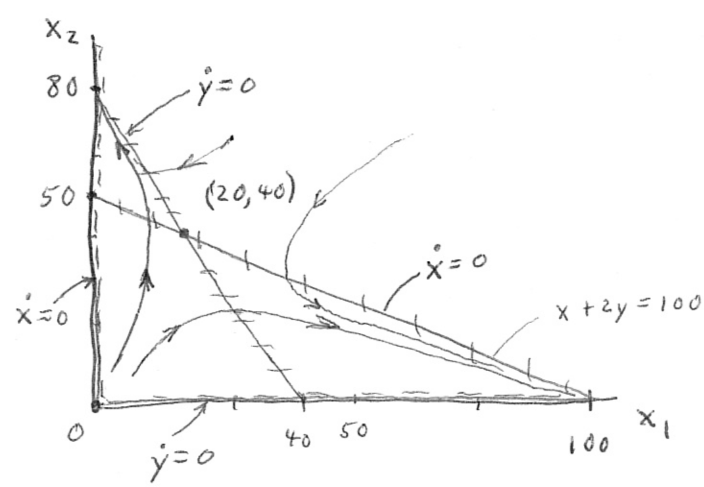


3. (20pts) Find the general solution to the following system of equations:

$$\frac{dx_1}{dt} = x_1 - 5\,x_2 + 7,$$
$$\frac{dx_2}{dt} = x_1 - x_2 + 3.$$

Sketch a phase portrait for this system.

Equil.

$x_{1e} - 5x_{2e} = -7$ $\qquad x_{2e} = 1$ $\qquad \begin{pmatrix} x_{1e} \\ x_{2e} \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$-(x_{1e} - x_{2e} = -3)$ $\qquad x_{1e} = -3+1 = -2$

$-4x_{2e} = -4$ $\qquad$ 4

$z_1(t) = x_1(t) + 2$ $\qquad \frac{dz_1}{dt} = z_1 - 5z_2$

$z_2(t) = x_2(t) - 1$ $\qquad \frac{dz_2}{dt} = z_1 - z_2$ $\qquad$ 2

$\dot{z} = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} z$ $\qquad \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 + 5 = \lambda^2 + 4 = 0$ $\qquad$ 2

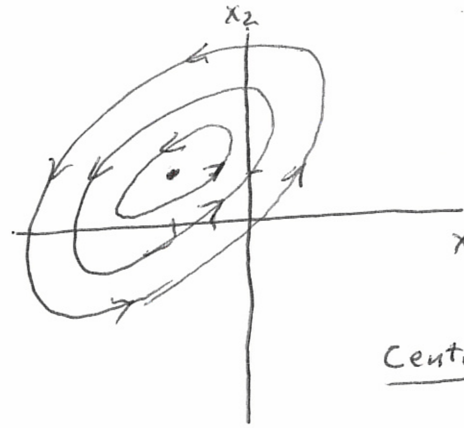
$\lambda = \pm 2i$

4 $\quad \lambda_1 = 2i$ $\qquad v_1 = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \vec{0}$

$\xi_1 + (-1-2i)\xi_2 = 0$

$z_1(t) = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}(\cos(2t) + i\sin(2x))$



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Center

$z(t) = c_1 \begin{pmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 \begin{pmatrix} 2\cos(2t) + \sin(2t) \\ \sin(2t) \end{pmatrix}$

4 $\quad x(t) = c_1 \begin{pmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 \begin{pmatrix} 2\cos(2t) + \sin(2t) \\ \sin(2t) \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$