February 28, 2018 Math 337 Name Key

Note: For full credit you must **show intermediate steps** in your calculations.

1. (27pts) Consider the following initial value problems. Find the unique solution and show how you obtained your solution.

a. $t\dfrac{dy}{dt} = 5t + 2y - 4,$ with $y(1) = 7.$

$\frac{dy}{dt} - \frac{2}{t}y = 5 - \frac{4}{t}$, $\mu(t) = exp\left(\int(-\frac{2}{t})dt\right) = exp(-2\ln|t|) = t^{-2}$

8

$\frac{d}{dt}(t^{-2}y) = \int(5t^{-2} - 4t^{-3})dt$

$t^{-2}y(t) = -5t^{-1} + 2t^{-2} + C$

$y(t) = -5t + 2 + Ct^2$

$y(1) = 7 = -5 + 2 + C \Rightarrow C = 10$

$y(t) =$ $10t^2 - 5t + 2$

b. $2y(4 + \sin(2t))\dfrac{dy}{dt} = 8t^3 - 2y^2\cos(2t),$ with $y(0) = -2.$

$2y^2\cos(2t) - 8t^3 + 2y(4 + \sin(2t))\frac{dy}{dt} = 0$

$\frac{\partial M}{\partial y} = 4y\cos(2t)$

$\frac{\partial N}{\partial t} = 4y\cos(2t)$ $\therefore$ exact

10

$\phi(t,y) = \int(2y^2\cos(2t) - 8t^3)dt = y^2\sin(2t) - 2t^4 + h(y)$

$\phi(t,y) = \int 2y(4 + \sin(2t))dy = 4y^2 + y^2\sin(2t) + k(t)$

$h(y) = 4y^2 + C$

$k(t) = -2t^4 + C$

$\phi(t,y) = y^2\sin(2t) - 2t^4 + 4y^2 + C = 0$, I.C. $\Rightarrow 16 + C = 0$ $C = -16$

$y^2(4 + \sin(2t)) = 16 + 2t^4$

$y(t) =$ $-\sqrt{\dfrac{16 + 2t^4}{4 + \sin(2t)}}$

c. $\dfrac{dy}{dt} = 2y - y^2e^{-3t},$ with $y(0) = 3.$

$-y^{-2}\left(\frac{dy}{dt} - 2y = -y^2e^{-3t}\right)$

$u = y^{1-2} = y^{-1}$

$\frac{du}{dt} = -y^{-2}\frac{dy}{dt}$

$\frac{du}{dt} + 2u = e^{-3t}$

9

$\frac{d}{dt}(e^{2t}u) = e^{-t}$

$e^{2t}u(t) = \frac{e^{2t}}{y(t)} = -e^{-t} + C$

$y(t) = \frac{e^{2t}}{C - e^{-t}}$

$y(0) = 3 = \frac{1}{C - 1}$

$\Rightarrow C = 4/3$

$y(t) =$ $\dfrac{e^{2t}}{4/3 - e^{-t}} = \dfrac{3}{4e^{-2t} - 3e^{-3t}}$

27

2. (20pts) Consider the initial value problem:

$$\frac{dy}{dt} = 2(1 + 2t)\sqrt{y}, \qquad y(0) = 1.$$

a. Solve this initial value problem.

$\int y^{-1/2} dy = \int 2(1+2t)dt$

$2y^{1/2} = 2(t+t^2) + 2C$

$y^{1/2} = t^2 + t + C$

$y(t) = (t^2+t+C)^2$

$y(0) = 1 = C^2$

8

$y(t) =$ $(t^2 + t + 1)^2$

b. Use Euler's method to simulate the solution for $t \in [0, 2]$ with a stepsize of $h = 0.5$. Write the specific Euler's formula that you use to compute your approximate solution y_n (with the specific h and $f(t, y)$), then show all of the steps, y_n, from $t = 0$ to $t = 2$.

2 $y_{n+1} =$ $y_n + 0.5\,(2(1+2t_n)\sqrt{y_n}) = y_n + (1+2t_n)\sqrt{y_n}$

8

t_n	y_n
$t_0 = 0$	$y_0 = 1$
$t_1 = 0.5$	$y_1 = 1 + (1+0)\sqrt{1} = 2$
$t_2 = 1$	$y_2 = 2 + (1+1)\sqrt{2} = 4.8284$
$t_3 = 1.5$	$y_3 = 4.8284 + (1+2)\sqrt{4.8284} = 11.4205$
$t_4 = 2$	$y_4 = 11.4205 + (1+3)\sqrt{11.4205} = 24.9382$

2 c. Compute the percent error between Euler approximation and the actual solution at $t = 2$.

$y(2) = (4+2+1)^2 = 49$

$100 \, \frac{(24.9382 - 49)}{49}$

Percent Error = -49.106%

20

3. (15pts) A good German Pilsner beer should be served at 40°F. At 11 AM, you grab some bottles that have been sitting in a warm garage (80°F) and plunge them into a bucket of ice water (32°F). At 11:30 AM, you find the temperature of the beer to be 65°F. Let $T(t)$ be the temperature of the beer, and assume the bottles of beer satisfy Newton's Law of Cooling,

$$\frac{dT}{dt} = -k(T(t) - T_e), = -k(T - 32)$$

where T_e is the temperature of the water in the ice bucket, t is in minutes, T is the temperature in °F, and k is the coefficient of heat transfer to be determined (to 4 significant figures.) At what time should you serve this beer.

$z(t) = T(t) - 32$, $\quad \frac{dz}{dt} = -kz$, $\quad z(0) = 80 - 32 = 48$

$z(t) = 48e^{-kt} \Rightarrow T(t) = 32 + 48e^{-kt}$

8 $T(30) = 32 + 48e^{-30k} = 65$ or $48e^{-30k} = 33 \Rightarrow e^{30k} = \frac{48}{33}$

$k = \frac{1}{30}\ln\left(\frac{48}{33}\right)$

$40 = 32 + 48e^{-0.01249t}$

$8 = 48e^{-0.01249t}$

$e^{0.01249t} = 6$

$t = \frac{\ln(6)}{0.01249}$

3 $k =$ 0.01249

$T(t) =$ $32 + 48e^{-0.01249t}$

3 Length of Time until properly cooled = 143.5 min

1 Clock Time for serving = 1:23.5 PM (Give the hour and minutes.)

15

4. (20pts) a. A population of animals is studied over a period of days. One group of researchers decides that the appropriate model describing this population satisfies the differential equation:

$$\frac{dP}{dt} = (0.05 - 0.00088\,t)P, \qquad P(0) = 64.$$

Solve this differential equation. Find when this model predicts that the population achieves a maximum and what that maximum population is.

$$\frac{dP}{dt} + (0.00088t - 0.05)P = 0$$

$$\mu(t) = e^{0.00044t^2 - 0.05t}$$

$$\frac{d}{dt}\left(e^{0.00044t^2 - 0.05t}P\right) = 0$$

$$P(t) = Ce^{0.05t - 0.00044t^2}$$

$$P(0) = C = 64$$

Max $\Rightarrow \frac{dP}{dt} = 0$

$0.00088t = 0.05$

$t = \frac{0.05}{0.00088} = 56.818$

5 $P(t) =$ $64e^{0.05t - 0.00044t^2}$

$P(56.818)$

2, 1 Time of Maximum = 56.818 days Maximum Population = 264.9

b. A second group of researchers thinks that a different model better describes the population of these animals. Their model satisfies the differential equation:

$$\frac{dP}{dt} = (0.3\,e^{-0.01t} - 0.1)P^{2/3}, \qquad P(0) = 64.$$

Solve this differential equation. Also, find when this model predicts that the population achieves a maximum and what that maximum population is.

$$\int P^{-2/3}dP = \int (0.3e^{-0.01t} - 0.1)\,dt$$

$$3P^{1/3} = -30e^{-0.01t} - 0.1t + C$$

$$P(t) = \left(\frac{C}{3} - 10e^{-0.01t} - \frac{0.1t}{3}\right)^3$$

$$P(0) = 64 = \left(\frac{C}{3} - 10\right)^3$$

$$4 = \frac{C}{3} - 10 \Rightarrow \frac{C}{3} = 14$$

Max $\frac{dP}{dt} = 0$

$0.3e^{-0.01t} = 0.1$

$e^{0.01t} = 3$

$t = 100\ln(3)$

8 $P(t) =$ $\left(14 - 10e^{-0.01t} - \frac{t}{30}\right)^3$

$P(109.86)$

3, 1 Time of Maximum = 109.86 days Maximum Population = 343.68

20

5. (18pts) a. In many population studies, the animals follow the logistic growth model. Suppose that along a certain river it is found that the population dynamics for a particular species of game fish satisfies the differential equation

$$\frac{dP}{dt} = 0.2P - 0.002P^2, \qquad \left(\frac{dP}{dt} - 0.2P = -0.002P^2\right) \cdot P^{-2}$$

where t is in years and P is the density of fish/100 m of river. Suppose that this river is stocked with an initial population of fish $P(0) = 10$ for each 100 m of river. Solve this initial value problem to determine the population of fish at any time after the river is stocked.

Bernoulli $\quad u = P^{1-2} = P^{-1} \qquad \frac{du}{dt} = -P^{-2}\frac{dP}{dt}$

$\frac{du}{dt} + 0.2u = 0.002 \qquad \mu(t) = e^{0.2t} \qquad \frac{d}{dt}\left(e^{0.2t}u\right) = 0.002e^{0.2t}$

$e^{0.2t}u(t) = e^{0.2t}/P(t) = 0.01e^{0.2t} + C \qquad P(t) = \frac{1}{0.01 + Ce^{-0.2t}}$

10

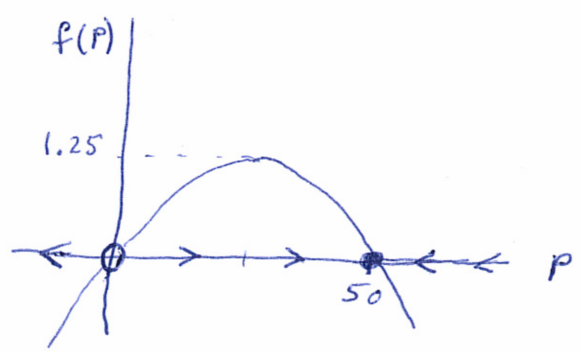
$P(t) = \underline{\dfrac{100}{1 + 9e^{-0.2t}}} \qquad P(0) = 10 = \frac{1}{0.01 + C} \Rightarrow C = 0.09$

b. It happens that this is a very popular species of game fish. Assume that h is the intensity of fishing with fish being caught and removed proportional the existing population with the model

$$\frac{dP}{dt} = 0.2P - 0.002P^2 - hP,$$

Let $h = 0.1$ and sketch a graph of the right hand side of the differential equation, then draw the phase portrait with arrows and circles (unstable(open circle); stable(filled circle)). Find any equilibria and determine their stability.

$$\frac{dP}{dt} = 0.2P - 0.002P^2 - 0.1P = 0.1P(1 - 0.02P)$$



4

4 List all equilibria (with stability), $P_e =$ 0 (unstable), 50 (stable)

18