Math 531 - Partial Differential Equations
Sturm-Liouville Problems
Part C

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Spring 2023

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Rayleigh Quotient

The **Sturm-Liouville Differential Equation** problem:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx} \right) + q(x)\phi + \lambda\sigma(x)\phi = 0.$$

Multiply by ϕ and integrate:

$$\int_a^b \left[\phi\frac{d}{dx}\left(p(x)\frac{d\phi}{dx} \right) + q(x)\phi^2 \right] dx + \lambda\int_a^b \phi^2\sigma(x)dx = 0.$$

The **eigenvalue** satisfies:

$$\lambda = -\frac{\int_a^b \left[\phi\frac{d}{dx}\left(p(x)\frac{d\phi}{dx} \right) + q(x)\phi^2 \right] dx}{\int_a^b \phi^2\sigma(x)dx}.$$

Rayleigh Quotient

Integrate the **eigenvalue** equation by parts:

$$\lambda = \frac{-\left. p\phi\frac{d\phi}{dx} \right|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q(x)\phi^2 \right] dx}{\int_a^b \phi^2\sigma(x)dx},$$

which is the **Rayleigh Quotient**.

The **eigenvalues** are nonnegative ($\lambda \geq 0$), if

1. $-\left. p\phi\frac{d\phi}{dx} \right|_a^b \geq 0,$

2. $q \leq 0.$

These conditions commonly hold for **Physical problems**, where $q \leq 0$ or *energy-absorbing*.

Minimization Principle

The **eigenvalue** satisfies:

Theorem (Minimization Principle)

*The minimum value of the **Rayleigh quotient** for all continuous functions satisfying the **BCs** (not necessarily the differential equation) is the **lowest eigenvalue**:*

$$\lambda = \min_u \frac{- pu\frac{du}{dx}\big|_a^b + \int_a^b \left[p\left(\frac{du}{dx}\right)^2 - q(x)u^2 \right] dx}{\int_a^b u^2 \sigma(x)dx},$$

This **minimum** occurs at $u = \phi_1$, the **lowest eigenfunction**.

Trial functions

Trial functions: Cannot test all ***continuous functions*** satisfying the **BCs**, but select ***trial functions***, u_T,

$$\lambda_1 \leq RQ[u_T] = \frac{- pu_T\frac{du_T}{dx}\big|_a^b + \int_a^b \left[p\left(\frac{du_T}{dx}\right)^2 - q(x)u_T^2 \right] dx}{\int_a^b u_T^2 \sigma(x)dx},$$

This provides an ***upper bound*** for λ_1.

Example: Consider the **Sturm-Liouville** problem:

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = 0 \quad \text{and} \quad \phi(1) = 0.$$

This example has an ***eigenvalue***, $\lambda_1 = \pi^2$, with an associated ***eigenfunction***, $\phi_1 = \sin(\pi x)$.

Trial functions

Example: We compute the **Rayleigh quotient** with **3** test functions, $u_1(x)$, $u_2(x)$, and $u_3(x)$:
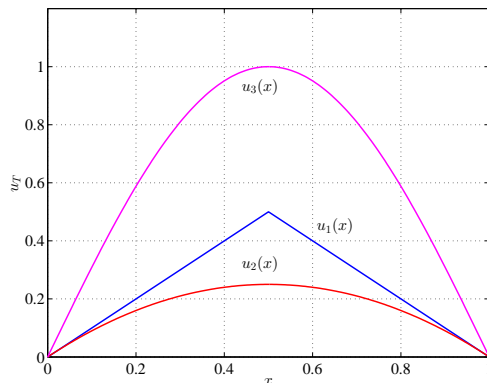
Tent function:

$$u_1(x) = \begin{cases} x, & x < \frac{1}{2}, \\ 1-x, & x \geq \frac{1}{2}. \end{cases}$$

Quadratic function:

$$u_2(x) = x - x^2.$$

Eigenfunction:

$$u_3(x) = \sin(\pi x).$$

We insert each of these functions into the **Rayleigh quotient**.

Trial functions

Example: The **Rayleigh quotient** with

$$u_1(x) = \begin{cases} x, & x < \frac{1}{2}, \\ 1-x, & x \geq \frac{1}{2}, \end{cases}$$

satisfies:

$$\begin{aligned} \lambda_1 \leq RQ[u_1] &= \frac{- u_1\frac{du_1}{dx}\big|_0^1 + \int_0^1 \left(\frac{du_1}{dx}\right)^2 dx}{\int_0^1 u_1^2 dx}, \\ &= \frac{\int_0^{1/2} dx + \int_{1/2}^1 dx}{\int_0^{1/2} x^2 dx + \int_{1/2}^1 (1-x)^2 dx}, \\ &= \frac{\frac{1}{2} + \frac{1}{2}}{\frac{1}{24} + \frac{1}{24}} = 12. \end{aligned}$$

Trial functions

Example: The **Rayleigh quotient** with $u_2(x) = x - x^2$ satisfies:

$$\lambda_1 \leq RQ[u_2] = \frac{-u_2 \frac{du_2}{dx}\big|_0^1 + \int_0^1 \left(\frac{du_2}{dx}\right)^2 dx}{\int_0^1 u_2^2 dx},$$

$$= \frac{\int_0^1 (1-2x)^2 dx}{\int_0^1 (x-x^2)^2 dx} = \frac{1 - 2 + \frac{4}{3}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{5}} = 10.$$

The **Rayleigh quotient** with $u_3(x) = \sin(\pi x)$ satisfies:

$$\lambda_1 \leq RQ[u_3] = \frac{-u_3 \frac{du_3}{dx}\big|_0^1 + \int_0^1 \left(\frac{du_3}{dx}\right)^2 dx}{\int_0^1 u_3^2 dx},$$

$$= \pi^2 \frac{\int_0^1 \cos^2(\pi x) dx}{\int_0^1 \sin^2(\pi x) dx}, = \frac{\pi^2 \frac{1}{2}}{\frac{1}{2}} = \pi^2 \approx 9.8696.$$

Rayleigh quotient

Proof: The proof of the **Rayleigh quotient** generally uses the **Calculus of Variations**, which cannot be developed here.

Our proof is based on *eigenfunction expansion*.

We assume u is a *continuous* function satisfying *homogeneous BCs*

Assuming *homogeneous BCs* gives the equivalent form for the **Rayleigh quotient**:

$$RQ[u] = \frac{-\int_a^b u L(u) dx}{\int_a^b u^2 \sigma dx},$$

where L is the **Sturm-Liouville operator**.

We take u expanded by the *eigenfunctions*

$$u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Rayleigh quotient

Proof (cont): Since L is a *linear operator*, we expect

$$L(u) = \sum_{n=1}^{\infty} a_n L(\phi_n(x)) = -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \phi_n(x),$$

where later we show the interchange of the summation and operator when u is *continuous* and satisfies *homogeneous BCs* of the *eigenfunctions*.

With different dummy summations, the **Rayleigh quotient** becomes

$$RQ[u] = \frac{\int_a^b \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m a_n \lambda_n \phi_m \phi_n \sigma\right) dx}{\int_a^b \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m a_n \phi_m \phi_n \sigma\right) dx}.$$

We interchange the summation and integration and use **orthogonality** to give

$$RQ[u] = \frac{\sum_{n=1}^{\infty} a_n^2 \lambda_n \int_a^b \phi_n^2 \sigma dx}{\sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma dx}.$$

Rayleigh quotient

Proof: The previous equation gives the exact expression for the **Rayleigh quotient** in terms of the generalized **Fourier coefficients** a_n of u. If λ_1 is the lowest *eigenvalue*, then we obtain:

$$RQ[u] \geq \frac{\lambda_1 \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma dx}{\sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma dx} = \lambda_1.$$

Note that equality holds only if $a_n = 0$ for $n > 1$, which gives the **minimization** result that $RQ[u] = \lambda_1$ for $u = a_1 \phi_1$.

The proof is easily extended to show that if $a_1 = 0$ for the *eigenfunction expansion* of u, then $RQ[u] = \lambda_2$ when $a_n = 0$ for $n > 2$ and $u = a_2 \phi_2$.

Thus, the **minimum** value for all continuous functions u that are *orthogonal to the lowest eigenfunction* and satisfy the *homogeneous BCs* is the next-to-lowest *eigenvalue*.

Rayleigh Quotient
Robin Boundary Conditions
Other Properties - Sturm-Liouville
Eigenvalue Equation
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Robin Boundary Conditions

Heat Equation with BC of Third Kind: Consider the **PDE**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with the **BCs**

$$u(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = -hu(L,t).$$

If $h > 0$, then this is a *physical problem* and the right endpoint represents **Newton's law of cooling** with an environmental temperature of $0°$.

Note: The problem solving below can be done equally well with the **String Equation**, $u_{tt} = c^2 u_{xx}$, where the right **BC** represents a restoring force for $h > 0$ and is called an *elastic BC*.

If $h < 0$, either problem is not physical, as the **heat equation** would be having heat constantly pumped into the rod, and the **string equation** has a destabilizing force on the right end.

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Robin Boundary Conditions

Separation of Variables: Let

$$u(x,t) = G(t)\phi(x),$$

then as before, the time dependent **ODEs** are

$$\textbf{Heat Flow:} \qquad \frac{dG}{dt} = -\lambda k G,$$

$$\textbf{Vibrating String:} \qquad \frac{d^2 G}{dt^2} = -\lambda c^2 G.$$

The **Sturm-Liouville problem** becomes:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(0) = 0 \quad \text{and} \quad \phi'(L) + h\phi(L) = 0,$$

where $h \geq 0$ is *physical* and $h < 0$ is *nonphysical*.

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Robin Boundary Conditions

Positive eigenvalues: Let $\lambda = \alpha^2 > 0$, then

$$\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

The **BC**, $\phi(0) = 0$, implies $c_1 = 0$.

The other **BC**, $\phi'(L) + h\phi(L) = 0$, implies that $c_2\left(\alpha\cos(\alpha L) + h\sin(\alpha L)\right) = 0$ or

$$\tan(\alpha L) = -\frac{\alpha}{h} = -\frac{\alpha L}{hL}.$$

This is a *transcendental equation* in α, which cannot be solved exactly.

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Robin Boundary Conditions

Eigenvalue equation is given by

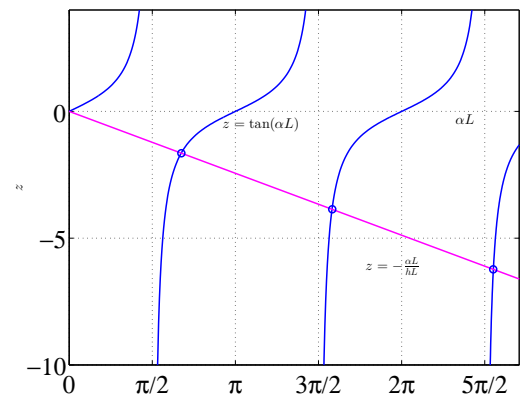$$\tan(\alpha L) = -\frac{\alpha L}{hL}, \qquad h > 0.$$

This equation can only be solved numerically, such as **Maple** or **MatLab**

This sketch is for the **physical** case, $h > 0$.

Visually, can see that asymptotically:

$$\alpha_n L \approx \left(n - \frac{1}{2}\right)\pi,$$

as $n \to \infty$

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Again the **eigenvalue equation** is given by

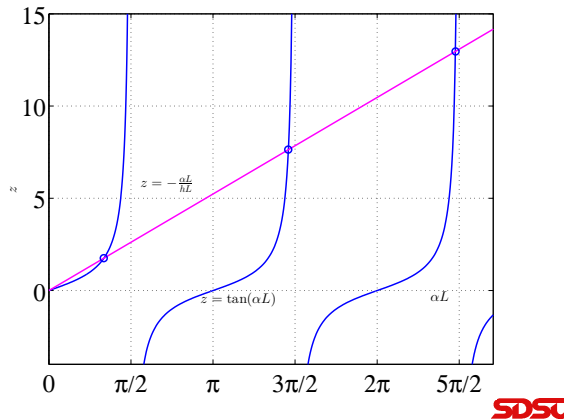$$\tan(\alpha L) = -\frac{\alpha L}{hL}, \qquad -1 < hL < 0.$$

This sketch is for the **nonphysical** case, $-1 < hL < 0$, which is 1 of 3 cases.

There is a lowest **eigenvalue**, $\lambda_1 < \frac{\pi}{2}$.

Asymptotically:

$$\alpha_n L \approx \left(n - \frac{1}{2}\right)\pi,$$

as $n \to \infty$

Rayleigh Quotient
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Other Properties - Sturm-Liouville

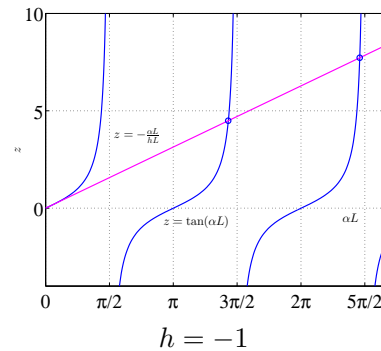**Eigenvalue Equation**
Zero and Negative Eigenvalue
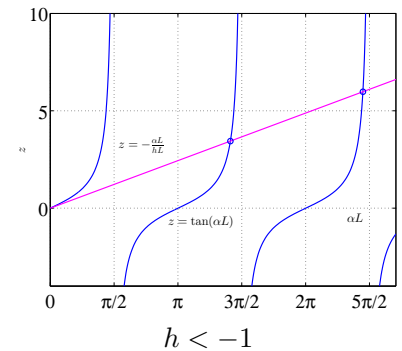Summary

Robin Boundary Conditions

There are two additional cases for the **nonphysical problem**, where

$$\tan(\alpha L) = -\frac{\alpha L}{hL}, \qquad hL = -1 \quad \text{or} \quad hL < -1.$$

In both cases, the first **positive eigenvalue** satisfies $\pi < \lambda < \frac{3\pi}{2}$.

$h = -1$ $h < -1$

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Robin Boundary Conditions

The **nonphysical problem** with $hL = -1$ has its first **positive eigenvalue**, $\alpha L \approx 4.49341$ ($\lambda = \alpha^2$).

Zero E.V.: Consider $\lambda = 0$, which gives the solution $\phi(x) = c_1 x + c_2$

The **BC** $\phi(0) = c_2 = 0$.

The other **BC**

$$\phi'(L) + h\phi(L) = c_1(1 + hL) = 0,$$

so if $hL = -1$, then $\lambda_0 = 0$ is an **eigenvalue** with associated **eigenfunction**,

$$\phi_0(x) = x.$$

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Negative E.V.: We don't expect negative **eigenvalues** for **physical problems**, as it produces an exponentially growing t-solution.

Suppose $\lambda = -\alpha^2 < 0$, so $\phi'' - \alpha^2 = 0$, which has the general solution:

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

The **BC** $\phi(0) = c_1 = 0$.

The remaining **BC** gives:

$$c_2 \left(\alpha \cosh(\alpha L) + h \sinh(\alpha L)\right) = 0,$$

which is nontrivial if

$$\tanh(\alpha L) = -\frac{\alpha}{h} = -\frac{\alpha L}{hL},$$

which is another **transcendental equation**.

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Robin Boundary Conditions
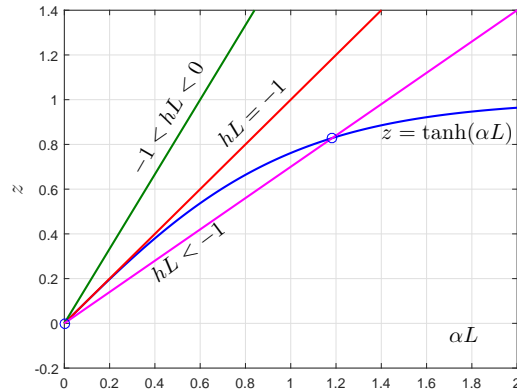
There are **4** cases to consider solving

$$\tanh(\alpha L) = -\frac{\alpha L}{hL}.$$

Physical case $(hL > 0)$ has a negative slope, so only intersects origin.

When $-1 < hL < 0$, only intersects origin.

When $hL = -1$, line is tangent to origin.

When $hL < -1$, there is a **unique positive eigenvalue**

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Robin Boundary Conditions - Physical Problem

Heat Equation: Consider the **PDE**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with the **BCs**

$$u(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = -hu(L,t), \quad h > 0,$$

and **ICs**

$$u(x,0) = f(x).$$

The **Sturm-Liouville problem** had *eigenvalues*, $\lambda_n = \alpha_n^2$, where $\alpha_n, \quad n = 1, 2, ...$ solves

$$\tan(\alpha_n L) = -\frac{\alpha_n L}{hL},$$

and corresponding *eigenfunctions*

$$\phi_n = \sin(\alpha_n x).$$

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Robin Boundary Conditions - Physical Problem

Heat Equation (cont): The time dependent solution is

$$G_n(t) = e^{-k\lambda_n t} = e^{-k\alpha_n^2 t}.$$

With the product solution, $u_n(x,t) = G_n(t)\phi_n(x)$, the **superposition principle** gives:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t}\sin(\alpha_n x),$$

where α_n satisfies $\tan(\alpha_n L) = -\frac{\alpha_n L}{hL}$.

The *generalized Fourier coefficients* satisfy:

$$A_n = \frac{\int_0^L f(x)\sin(\alpha_n x)dx}{\int_0^L \sin^2(\alpha_n x)dx}.$$

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Robin Boundary Conditions - Physical Problem

Heat Equation (cont): However, with $\sin(\alpha_n L) = -\frac{\alpha_n}{h}\cos(\alpha_n L)$

$$\int_0^L \sin^2(\alpha_n x)dx = \frac{2\alpha_n L - \sin(2\alpha_n L)}{4\alpha_n} = \frac{Lh + \cos^2(\alpha_n L)}{2h}.$$

Thus, the *generalized Fourier coefficients* satisfy:

$$A_n = \frac{2h\int_0^L f(x)\sin(\alpha_n x)dx}{Lh + \cos^2(\alpha_n L)},$$

and the temperature in the rod is given by

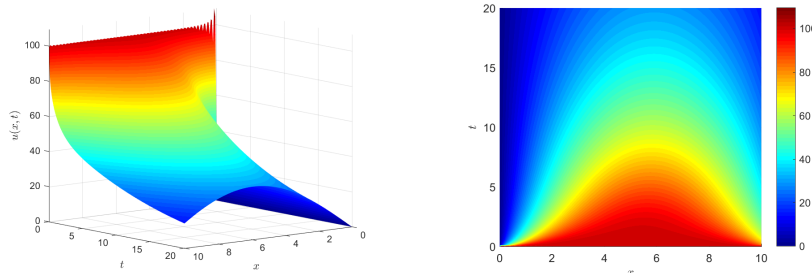$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t}\sin(\alpha_n x).$$

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Robin Boundary Conditions - Physical Problem

Take $L = 10$, $k = 1$, and $h = 0.5$ and suppose $f(x) = 100$ for $0 \le x \le 10$. The Fourier coefficients are readily found:

$$A_n = \frac{200h\left(1 - \cos(\alpha_n L)\right)}{\alpha_n\left(Lh + \cos^2(\alpha_n L)\right)}.$$

Solution with 100 terms.

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Robin Boundary Conditions - Physical Problem

```matlab
 1  % Solutions to the heat flow equation
 2  % on one-dimensional rod length L
 3  % Right end with Robin Condition
 4  format compact;
 5  L = 10;                % width of plate
 6  Temp = 100;            % Constant temperature of ...
       rod, initially
 7  tfin = 20;             % final time
 8  k = 1;                 % heat coef of the medium
 9  h = 0.5;               % Newton cooling constant
10  NptsX=151;             % number of x pts
11  NptsT=151;             % number of t pts
12  Nf=100;                % number of Fourier terms
13  x=linspace(0,L,NptsX);
14  t=linspace(0,tfin,NptsT);
15  [X,T]=meshgrid(x,t);
```

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Robin Boundary Conditions - Physical Problem

```matlab
17  figure(1)
18  clf
19  a = zeros(1,Nf);
20  b = zeros(1,Nf);
21  U = zeros(NptsT,NptsX);
22  z0 = 2.7;
23  for n=1:Nf
24      z0 = fsolve(@(x) h*L*sin(x)+x*cos(x),z0);
25      a(n) = z0/L;
26      b(n)=(2*Temp*h/(a(n)*(L*h+(cos(a(n)*L))^2)))...
27          *(1-cos(a(n)*L));   % Fourier coefficients
28      Un=b(n)*exp(-k*(a(n))^2*T).*sin(a(n)*X);   % ...
            Temperature(n)
29      U=U+Un;
30      z0 = z0 + pi;
31  end
```

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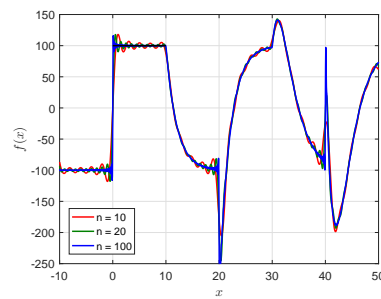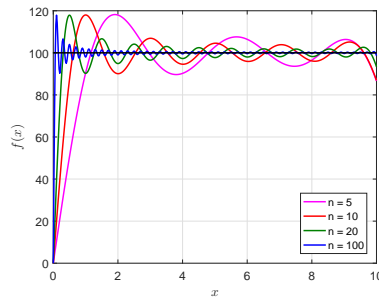## Robin Boundary Conditions - Physical Problem

```matlab
32  set(gca,'FontSize',[12]);
33  surf(X,T,U);
34  shading interp
35  colormap(jet)
36  xlabel('$x$','Fontsize',12,'interpreter','latex');
37  ylabel('$t$','Fontsize',12,'interpreter','latex');
38  zlabel('$u(x,t)$','Fontsize',12,'interpreter','latex');
39  axis tight
40  view([141 10])
```

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Fourier Series - BC 3rd Kind

The solution of the **Heat Equation** with **Robin BCs** used the Fourier expansion of $f(x) = 100$ with the eigenfunctions, $\phi_n = \sin(\alpha_n x)$. Below are graphs showing the eigenfunction expansion.

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Fourier Series - BC 3rd Kind

```matlab
1   % Fourier series
2   format compact;
3   L = 10;                 % width of plate
4   Temp = 100;             % Constant temperature of ...
        rod, initially
5   h = 0.5;                % Newton cooling constant
6   NptsX=500;              % number of x pts
7   Nf=100;                 % number of Fourier terms
8   X=linspace(0,L,NptsX);
9   a = zeros(1,Nf);
10  b = zeros(1,Nf);
11  U = zeros(1,NptsX);
12  U1 = zeros(1,NptsX);
13  U2 = zeros(1,NptsX);
14  U3 = zeros(1,NptsX);
15  z0 = 2.7;
```

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Fourier Series - BC 3rd Kind

```matlab
16  for n=1:Nf
17      z0 = fsolve(@(x) h*L*sin(x)+x*cos(x),z0);
18      a(n) = z0/L;
19      b(n)=(2*Temp*h/(a(n)*(L*h+(cos(a(n)*L))^2)))...
20          *(1-cos(a(n)*L));   % Fourier coefficients
21      Un = b(n)*sin(a(n)*X);  % Temperature(n)
22      U = U+Un;
23      if (n <= 5)
24          U1 = U1+Un;
25      end
26      if (n <= 10)
27          U2 = U2+Un;
28      end
29      if (n <= 20)
30          U3 = U3+Un;
31      end
32      z0 = z0 + pi;
33  end
```

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Fourier Series - BC 3rd Kind

```matlab
34  plot(X,U1,'m-','LineWidth',1.5);
35  hold on
36  plot(X,U2,'r-','LineWidth',1.5);
37  plot(X,U3,'-','Color',[0 0.5 0],'LineWidth',1.5);
38  plot(X,U,'b-','LineWidth',1.5);
39  plot([0 10],[100 100],'k-','LineWidth',1.5);
40  grid;
41  legend('n = 5','n = 10','n = 20','n = 100',...
42      'location','southeast');
43  xlim([0 10]);
44  ylim([0 120]);
45  xlabel('$x$','Fontsize',12,'interpreter','latex');
46  ylabel('$f(x)$','Fontsize',12,'interpreter','latex');
47  set(gca,'FontSize',[12]);
```

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Robin Boundary Conditions - Non-Physical Problem

Heat Equation with **Non-Physical BCs** satisfies:

PDE: $u_t = ku_{xx}$, **BC**: $u(0, t) = 0$,

IC: $u(x, 0) = f(x)$, $u_x(L, t) = -hu(L, t)$ with $h < 0$.

For $-1 < h < 0$, the **Sturm-Liouville problem** is the same as the **physical problem** with *eigenvalues*, $\lambda_n = \alpha_n^2$, where $\alpha_n, \ n = 1, 2, ...$ solves $\tan(\alpha_n L) = -\frac{\alpha_n L}{hL}$, and corresponding *eigenfunctions* are

$$\phi_n = \sin(\alpha_n x).$$

The solution satisfies:

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin(\alpha_n x),$$

with the same generalized Fourier coefficients as for the **physical problem**.

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Summary

Robin Boundary Conditions - Non-Physical Problem

Heat Equation with **Non-Physical BCs** and $h = -1$ has $\lambda_0 = 0$ with the eigenfunction $\phi_0(x) = x$, so the solution becomes:

$$u(x, t) = A_0 x + \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin(\alpha_n x),$$

with A_n as before for $n = 1, 2, ...$ and

$$A_0 = \frac{3}{L^3} \int_0^L x f(x) dx.$$

If $h < -1$ and β_1 solves $\tanh(\beta_1 L) = -\frac{\beta_1}{h}$, then there is the additional eigenfunction $\phi_{-1}(x) = \sinh(\beta_1 x)$, so the solution becomes:

$$u(x, t) = A_{-1} e^{k\beta_1^2 t} \sinh(\beta_1 x) + \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin(\alpha_n x),$$

with A_n as before for $n = 1, 2, ...$ and

$$A_{-1} = \frac{2\beta_1 \int_0^L f(x) \sinh(\beta_1 x) dx}{\cosh(\beta_1 L) \sinh(\beta_1 L) - \beta_1 L}.$$

Rayleigh Quotient
Robin Boundary Conditions
Other Properties - Sturm-Liouville

Eigenvalue Equation
Zero and Negative Eigenvalue
Summary

Robin Boundary Conditions - Physical Problem

Heat Equation with $h = 0$ (insulated right end) satisfies:

PDE: $u_t = ku_{xx}$, **BC**: $u(0, t) = 0$,

IC: $u(x, 0) = f(x)$, $u_x(L, t) = 0$.

This problem is solved in the normal manner as before, and it is easy to see that the *eigenvalues*, $\lambda_n = \frac{\left(n - \frac{1}{2}\right)^2 \pi^2}{L^2}$, with corresponding *eigenfunctions* are

$$\phi_n = \sin\left(\frac{\left(n - \frac{1}{2}\right)\pi x}{L}\right).$$

The solution satisfies:

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n t} \sin\left(\frac{\left(n - \frac{1}{2}\right)\pi x}{L}\right),$$

with similar Fourier coefficients to our original **Heat problem**.

Eigenvalue Asymptotic Behavior

Examine the **Sturm-Liouville eigenvalue problem** in the form

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

The *eigenvalues* generally must be computed numerically.

There is a number of people working on details of these problems, so the scope of this problem is beyond this course. (See Mark Dunster)

Interpret this problem like a **spring-mass** problem for large λ, where x is time and ϕ is position.

- $p(x)$ acts like the mass.
- For λ large, $-\lambda\sigma(x)\phi$ acts like a restoring force
- This solution rapidly oscillates

Eigenvalue Asymptotic Behavior

With large λ, the solution oscillates rapidly over a few periods, so can approximate the coefficients as constants.

Thus, the DE is approximated near any point x_0 by

$$p(x_0)\frac{d^2\phi}{dx^2} + \lambda\sigma(x_0)\phi \approx 0,$$

which is like a standard **spring-mass** problem.

It follows that the frequency is approximated by

$$\omega = \sqrt{\frac{\lambda\sigma(x_0)}{p(x_0)}}$$

Eigenvalue Asymptotic Behavior

The **amplitude** and **frequency** are slow varying, so

$$\phi(x) = A(x)\cos(\psi(x)).$$

With Taylor series, we write

$$\phi(x) = A(x)\cos[\psi(x_0) + \psi'(x_0)(x - x_0) + ...],$$

so the **local frequency** is $\psi'(x_0)$, where

$$\psi'(x_0) = \lambda^{1/2}\left(\frac{\sigma(x_0)}{p(x_0)}\right)^{1/2}.$$

Eigenvalue Asymptotic Behavior

Integrating $\psi'(x_0)$ gives the correct phase

$$\psi(x) = \lambda^{1/2}\int^x\left(\frac{\sigma(x_0)}{p(x_0)}\right)^{1/2}dx_0.$$

It can be shown (beyond this class) that the independent solutions are approximated for large λ by

$$\phi(x) \approx (\sigma p)^{-1/4}exp\left[\pm i\lambda^{1/2}\int^x\left(\frac{\sigma}{p}\right)^{1/2}dx_0\right].$$

If $\phi(0) = 0$, then the **eigenfunction** can be approximated by

$$\phi(x) = (\sigma p)^{-1/4}\sin\left(\lambda^{1/2}\int^x\left(\frac{\sigma}{p}\right)^{1/2}dx_0\right) + ...$$
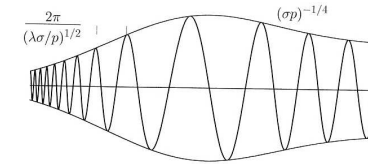
If the second BC is $\phi(L) = 0$, then

$$\lambda^{1/2}\int_0^L\left(\frac{\sigma}{p}\right)^{1/2}dx_0 \approx n\pi \quad \text{or} \quad \lambda \approx \left[\frac{n\pi}{\int_0^L\left(\frac{\sigma}{p}\right)^{1/2}dx_0}\right]^2.$$

Eigenvalue Asymptotic Behavior

Example: Consider the **eigenvalue problem**

$$\frac{d^2\phi}{dx^2} + \lambda(1 + x)\phi = 0,$$

with **BCs** $\phi(0) = 0$ and $\phi(1) = 0$.

Our approximation gives:

$$\lambda \approx \left[\frac{n\pi}{\int_0^1(1 + x_0)^{1/2}dx_0}\right]^2 = \frac{n^2\pi^2}{\left[\frac{2}{3}(1 + x_0)^{3/2}\big|_0^1\right]^2} = \frac{n^2\pi^2}{\frac{4}{9}(2^{3/2} - 1)^2}.$$

n	Numerical	Formula
1	6.5484	6.6424
2	26.4649	26.5697
3	59.6742	59.7819
4	106.1700	106.2789
5	165.9513	165.0607
6	239.0177	239.1275
7	325.3691	325.4790

Approximation Properties

We claimed that any *piecewise smooth function*, $f(x)$, can be represented by the *generalized Fourier series* of *eigenfunctions*:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

By *orthogonality with weight* $\sigma(x)$ of the eigenfunctions

$$a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)dx}{\int_a^b \phi_n^2(x)\sigma(x)dx}.$$

Suppose we use a finite expansion,

$$f(x) \approx \sum_{n=1}^{M} \alpha_n \phi_n(x).$$

How do we choose α_n to obtain the best approximation?

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Approximation Properties

How do we define the "best approximation?"

> **Definition (Mean-Square Deviation)**
>
> The standard measure of **Error** is the **mean-square deviation**, which is given by:
>
> $$E = \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \sigma(x)dx.$$

This deviation uses the weighting function, $\sigma(x)$.

It penalizes heavily for a large deviation on a small interval.

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Approximation Properties

The best approximation solves the system:

$$\frac{\partial E}{\partial \alpha_i} = 0, \qquad i = 1, 2, ..., M.$$

or

$$0 = \frac{\partial E}{\partial \alpha_i} = -2 \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right] \phi_i(x)\sigma(x)dx, \qquad i = 1, 2, ..., M.$$

This would be complicated, except that we have mutual *orthogonality* of the $\phi_i(x)$'s, so

$$\int_a^b f(x)\phi_i(x)\sigma(x)dx = \alpha_i \int_a^b \phi_i^2(x)\sigma(x)dx.$$

Solving this system for α_i gives the α_i as the *generalized Fourier coefficients*.

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Approximation Properties

An alternate proof of this result shows that the *minimum error* is:

$$E = \int_a^b f^2 \sigma dx - \sum_{n=1}^{M} \alpha_n^2 \int_a^b \phi_n^2 \sigma dx.$$

This equation shows that as M increases, the **error** decreases.

> **Definition (Bessel's Inequality)**
>
> Since $E \geq 0$,
>
> $$\int_a^b f^2 \sigma dx \geq \sum_{n=1}^{M} \alpha_n^2 \int_a^b \phi_n^2 \sigma dx.$$

More importantly, any *Sturm-Liouville eigenvalue problem* has an *eigenfunction expansion* of $f(x)$, which converges in the *mean* to $f(x)$.

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Approximation Properties

The ***convergence in mean*** implies that

$$\lim_{M\to\infty} E = 0,$$

which gives the following:

Definition (Parseval's Equality)

Since $E \geq 0$,

$$\int_a^b f^2 \sigma dx = \sum_{n=1}^{\infty} \alpha_n^2 \int_a^b \phi_n^2 \sigma dx.$$

This inequality is a ***generalization of the Pythagorean theorem***, which important in showing **completeness**.

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