Math 531 - Partial Differential Equations

Sturm-Liouville Problems
Part B

Joseph M. Mahaffy,
⟨jmahaffy@mail.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

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Outline

Regular Sturm-Liouville Problem

The **Regular Sturm-Liouville** problem satisfies:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0,$$

with the ***homogeneous BCs***:

$$\begin{aligned} \beta_1\phi(a) + \beta_2\phi'(a) &= 0, \\ \beta_3\phi(b) + \beta_4\phi'(b) &= 0, \end{aligned}$$

where (i) β_i are real, (ii) The functions $p(x)$, $q(x)$, and $\sigma(x)$ are real, continuous functions for $x \in [a, b]$ with $p(x) > 0$ and $\sigma(x) > 0$.

The following theorems will NOT be proved:

1. There are infinitely many ***eigenvalues***.
2. Any ***piecewise smooth*** function can be expanded by the ***eigenfunctions***.
3. Each succeeding ***eigenfunction*** has an additional zero.

Linear Operator

Linear Operator: Let L be the ***linear differential operator***:

$$\begin{aligned} L &= \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x), \\ L(y) &= \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y. \end{aligned}$$

The **Sturm-Liouville differential equation** is written:

$$L(\phi) + \lambda\sigma(x)\phi = 0,$$

where λ is an ***eigenvalue*** and ϕ is an ***eigenfunction***.

Note: The **Linear Operator** does NOT have to act on an ***eigenvalue problem***.

Lagrange's Identity

Theorem (Lagrange's Identity)

*Let L be the **Linear Operator**:*

$$L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x).$$

The following formula:

$$uL(v) - vL(u) = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right],$$

*is known as the **differential form of Lagrange's identity**.*

Linear Operator

Proof: This identity is readily shown

$$\begin{aligned} uL(v) - vL(u) &= u\left[\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + q(x)v\right] - v\left[\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u\right], \\ &= u\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p(x)\frac{du}{dx}\right). \end{aligned}$$

However, from the product rule we see that

$$\begin{aligned} \frac{d}{dx}\left(u\left(p\frac{dv}{dx}\right)\right) &= u\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + \frac{du}{dx}\cdot p\frac{dv}{dx}, \\ \frac{d}{dx}\left(v\left(p\frac{du}{dx}\right)\right) &= v\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + \frac{dv}{dx}\cdot p\frac{du}{dx}. \end{aligned}$$

Subtracting these equations, we can insert into the expression for $uL(v) - vL(u)$ to obtain **Lagrange's identity**:

$$uL(v) - vL(u) = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]. \qquad q.e.d.$$

Green's Formula

Lagrange's identity relates to the first part of the ***linear differential operator*** from the Sturm-Liouville problem.

Theorem (Green's Formula)

The integration of ***Lagrange's identity*** *give's* ***Green's formula****:*

$$\int_a^b [uL(v) - vL(u)]dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$$

for u and v continuously differentiable for $x \in (a, b)$.

This ***linear differential operator*** has important properties when there are ***homogeneous BCs***.

Self-adjoint

Suppose that u and v are any two functions with the additional restriction that the boundary terms vanish:

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\bigg|_a^b = 0.$$

Theorem (Self-adjoint)

*If u and v are any two functions satisfying the same set of **homogeneous BCs** of the **regular Sturm-Liouville problem**, then*

$$\int_a^b [uL(v) - vL(u)]dx = 0.$$

*The **linear operator** L satisfying this condition is **self-adjoint**.*

Self-adjoint

The theorem for **self-adjointness** of the **linear operator** L extends to other **Sturm-Liouville problems**.

1. HW Exercises examine some **regular Sturm-Liouville problems**.
2. It is easy to show for **periodic BCs**.
3. Also, easy to show if the **Sturm-Liouville problem** is singular at $x = 0$ and has $p(0) = 0$ with $\phi(0)$ bounded.

Orthogonality of Eigenfunctions

Let λ_n and λ_m be distinct ***eigenvalues*** with corresponding ***eigenfunctions*** ϕ_n and ϕ_m, so

$$\begin{aligned} L(\phi_n) + \lambda_n \sigma(x) \phi_n &= 0, \\ L(\phi_m) + \lambda_m \sigma(x) \phi_m &= 0. \end{aligned}$$

It follows that

$$\int_a^b [\phi_m L(\phi_n) - \phi_n L(\phi_m)] dx = (\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \sigma(x) dx$$

By **Green's Formula**:

$$(\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \sigma(x) dx = p(x) \left(\phi_m \frac{d\phi_n}{dx} - \phi_n \frac{d\phi_m}{dx} \right) \Big|_a^b .$$

Orthogonality of Eigenfunctions

From before

$$(\lambda_m - \lambda_n)\int_a^b \phi_m\phi_n\sigma(x)dx = p(x)\left(\phi_m\frac{d\phi_n}{dx} - \phi_n\frac{d\phi_m}{dx}\right)\Bigg|_a^b,$$

so if ϕ_m and ϕ_n satisfy the same ***homogeneous BCs*** the right hand side is **zero**.

Thus,

$$(\lambda_m - \lambda_n)\int_a^b \phi_m\phi_n\sigma(x)dx = 0,$$

which says that ***eigenfunctions*** corresponding to different ***eigenvalues*** are **orthogonal** with respect to the weighting function, $\sigma(x)$.

Real Eigenvalues

Suppose that an ***eigenvalue***, λ, is **complex** and it has a corresponding ***eigenfunction***, $\phi(x)$,

$$L(\phi) + \lambda\sigma\phi = 0.$$

Take the complex conjugate, so

$$\overline{L(\phi) + \lambda\sigma\phi} = \overline{L(\phi)} + \overline{\lambda}\sigma\overline{\phi} = 0.$$

However, $\overline{L(\phi)} = L(\overline{\phi})$, since the coefficients of L are real, so

$$L(\overline{\phi}) + \overline{\lambda}\sigma\overline{\phi} = 0.$$

Thus, if λ is a **complex** ***eigenvalue*** with corresponding ***eigenfunction*** ϕ, then $\overline{\lambda}$ is also an ***eigenvalue*** with ***eigenfunction*** $\overline{\phi}$.

Real Eigenvalues

By our **orthogonality theorem**,

$$(\lambda - \overline{\lambda}) \int_a^b \phi\overline{\phi}\sigma dx = 0.$$

However, $\phi\overline{\phi} = |\phi|^2 \geq 0$ and $\sigma > 0$.

Thus, the integral is **zero** if and only if $\phi(x) \equiv 0$, which is not an ***eigenfunction***, or $\lambda - \overline{\lambda} = 0$, which implies λ is real.

Thus, ***eigenvalues*** of a **Sturm-Liouville problem** are **real**.

Uniqueness

Suppose that ϕ_1 and ϕ_2 are ***eigenfunctions*** corresponding to λ, so

$$L(\phi_1) + \lambda\sigma\phi_1 = 0 \qquad \text{and} \qquad L(\phi_2) + \lambda\sigma\phi_2 = 0.$$

Since λ is the same,

$$\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = 0.$$

Lagrange's identity implies:

$$\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = \frac{d}{dx}\left[p\left(\phi_2\frac{d\phi_1}{dx} - \phi_1\frac{d\phi_2}{dx}\right)\right] = 0.$$

Therefore, $p\left(\phi_2\frac{d\phi_1}{dx} - \phi_1\frac{d\phi_2}{dx}\right)$ is a constant.

Uniqueness

Using the **Lagrange's identity**, we obtained

$$p\left(\phi_2\frac{d\phi_1}{dx} - \phi_1\frac{d\phi_2}{dx}\right) = C.$$

The constant $C = 0$, if we have **regular Sturm-Liouville problem** with at least one ***homogeneous BC***.

It follows that

$$\phi_2\frac{d\phi_1}{dx} - \phi_1\frac{d\phi_2}{dx} = 0 \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{\phi_2}{\phi_1}\right) = 0.$$

Thus, $\phi_2(x) = c\phi_1(x)$ with these BCs, so the ***eigenfunction*** is **unique**.

Non-Uniqueness

Earlier we showed that ***periodic BCs*** give both sine and cosine ***eigenfunctions***, so this case does **NOT** produce ***unique eigenfunctions*** for a given ***eigenvalue***.

Without ***homogeneous BCs***, the ***eigenvalue***, λ may not have a ***unique eigenfunction***.

Non-uniqueness can create problems with **orthogonality** for a given ***eigenvalue***.

The **Gram-Schmidt** process can be applied to create an ***orthogonal set*** of ***eigenfunctions***.

Recall that $\lambda_m \neq \lambda_n$ always produces ***orthogonal eigenfunctions***.