

# Math 531 - Partial Differential Equations

## Sturm-Liouville Problems

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# Outline

- 1 Heat Problems
- 2 Sturm-Liouville Eigenvalue Problem
  - Theorems
  - Nonuniform Rod

# Heat in Nonuniform Rod

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**Heat Flow in Nonuniform Rod:** Suppose that the *specific heat*,  $c(x)$ , *density*,  $\rho(x)$ , and *thermal conductivity*,  $K_0(x)$ , all depend on the spatial variable  $x$

Suppose that the *heat source*  $Q(x, t) = \alpha(x)u(x, t)$  satisfies **Newton's Law of Cooling**, which is proportional to heat in the bar (with environmental temperature being zero)

From before, this gives the **Heat Equation**:

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right) + \alpha u,$$

which is *homogeneous*

## Heat in Nonuniform Rod

**Heat Flow in Nonuniform Rod (cont):** Apply *separation of variables*,  $u(x, t) = \phi(x)h(t)$ , to the **PDE** and rearrange to

$$\frac{h'}{h} = \frac{1}{c\rho\phi} \frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right) + \frac{\alpha}{c\rho} = -\lambda.$$

The **differential equation** in  $x$  is

$$\frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right) + \alpha\phi + \lambda c\rho\phi = 0.$$

This is a **Sturm-Liouville Problem**, if there are *homogeneous BCs*

Solution to this *differential equation* may be difficult to find.

# Circularly Symmetric Heat Flow

1

**Circularly Symmetric Heat Flow:** Consider a circularly symmetric region with a uniform material, so  $k = \frac{K_0}{c\rho}$ , the **Heat Equation** is

$$\frac{\partial u}{\partial t} = k \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right).$$

Apply *separation of variables*,  $u(r, t) = \phi(r)h(t)$ , to the **PDE** and rearrange to

$$\frac{h'}{kh} = \frac{1}{r\phi} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) = -\lambda.$$

The **differential equation** in  $r$  is

$$\frac{d}{dr} \left( r \frac{d\phi}{dr} \right) + \lambda r \phi = 0.$$

## Circularly Symmetric Heat Flow

The **Sturm-Liouville Problem** in  $r$  is

$$\frac{d}{dr} \left( r \frac{d\phi}{dr} \right) + \lambda r \phi = 0,$$

if there are *homogeneous BCs*

For an *annulus*, the *homogeneous BCs* are

$$u(a, t) = 0 \quad \text{and} \quad u(b, t) = 0.$$

For a *circular region*, the *homogeneous BCs* are  $u(a, t) = 0$  and a *singularity condition*

$$|u(0, t)| < +\infty.$$

# Sturm-Liouville Eigenvalue Problem

The general Sturm-Liouville differential equation:

$$\frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + q(x)\phi + \lambda\sigma(x)\phi = 0,$$

where  $\lambda$  is an eigenvalue,  $a < x < b$ .

Examples to date are as follows:

- ❶ If  $p(x) = \sigma(x) = 1$  and  $q(x) = 0$ , then

$$\phi'' + \lambda\phi = 0.$$

- ❷ **Nonuniform heat flow:**  $K_0 = p(x)$ ,  $c\rho = \sigma(x)$ , and  $\alpha = q(x)$ ,

$$\frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right) + \alpha\phi + \lambda c\rho\phi = 0.$$

- ❸ **Circular heat flow:**  $p(r) = r$ ,  $\sigma(r) = r$ , and  $q(r) = 0$ ,

$$\frac{d}{dr} \left( r \frac{d\phi}{dr} \right) + \lambda r\phi = 0.$$

## Sturm-Liouville Eigenvalue Problem

The Sturm-Liouville eigenvalue problem with eigenvalue  $\lambda$  is:

$$\frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + q(x)\phi + \lambda\sigma(x)\phi = 0,$$

and requires *homogeneous BCs*

BCs	Heat Eqn	String Eqn	Type
$\phi = 0$	Ends zero Temp	Ends fixed	Dirichlet
$\phi' = 0$	Ends insulated	Ends free	Neumann
$\phi' = \pm h\phi$	Newton's cooling	Elastic boundary	Robin
$\phi(-L) = \phi(L)$ $\phi'(-L) = \phi'(L)$	Perfect thermal contact		Periodic
$ \phi(0)  < \infty$	Bounded Temp		Singularity



# Regular Sturm-Liouville Eigenvalue Problem

Consider the *second order differential equation*:

$$\frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \quad a < x < b.$$

The *homogeneous BCs* are:

$$\begin{aligned}\beta_1\phi(a) + \beta_2\phi'(a) &= 0, \\ \beta_3\phi(b) + \beta_4\phi'(b) &= 0,\end{aligned}$$

which exclude *periodic* and *singular BCs*.

The following conditions hold:

- $\beta_i$  are real
- The functions  $p(x)$ ,  $q(x)$ , and  $\sigma(x)$  are continuous and real for  $x \in [a, b]$  (including the endpoints)
- $p(x) > 0$  and  $\sigma(x) > 0$  for  $x \in [a, b]$  (including the endpoints)

# Important Theorems

**Important Theorems:** State and later prove some.

- 1 All *eigenvalues* are real.
- 2 There exist infinitely many *eigenvalues*,  $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$ 
  - a There is a smallest *eigenvalue*, denoted  $\lambda_1$ .
  - b There is not a largest *eigenvalue*, i.e.,  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ .
- 3 Corresponding to each *eigenvalue*,  $\lambda_n$ , there is an *eigenfunction*,  $\phi_n(x)$ , and  $\phi_n(x)$  has exactly  $n - 1$  zeros for  $x \in (a, b)$ .
- 4 The *eigenfunctions*,  $\phi_n(x)$ , form a **complete set**, meaning that any *piecewise smooth* function  $f(x)$  can be represented by a generalized **Fourier series**:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

Furthermore, the infinite series converges to  $[f(x^+) + f(x^-)]/2$  for all  $x \in (a, b)$  (with appropriate  $a_n$ )

# Important Theorems

**Important Theorems:** State and later prove some.

- 5 **Eigenfunctions** corresponding to different **eigenvalues** are **orthogonal** relative to the weight function,  $\sigma(x)$ ,

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx = 0, \quad \text{if } \lambda_n \neq \lambda_m.$$

- 6 Any **eigenvalue** can be related to its **eigenfunction** by the **Rayleigh quotient**

$$\lambda = \frac{-p(x)\phi(x)\phi'(x)\Big|_a^b + \int_a^b \left[ p(x) \left( \frac{d\phi}{dx} \right)^2 - q(x)\phi^2(x) \right] dx}{\int_a^b \phi^2(x)\sigma(x)dx},$$

where the **BCs** may simplify this expression.

# Example

**Example:** Consider the *Sturm-Liouville eigenvalue problem*:

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

We previously found the *eigenvalues*,  $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ , with *eigenfunctions*,  $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$  for  $n = 1, 2, \dots$

- 1 Found real eigenvalues, must establish not complex.
- 2 Smallest eigenvalue is  $\lambda_1 = \left(\frac{\pi}{L}\right)^2$ , and clearly  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ .
- 3 Easily seen that  $\phi_n(x)$  has  $n - 1$  zeros for  $x \in (0, L)$ .
- 4 Established **Fourier series** for this SL Problem, and showed *orthogonality* of  $\phi_n(x)$ .
- 5 The *Rayleigh quotient* simplifies to

$$\lambda = \frac{\int_0^L (\phi'(x))^2 dx}{\int_0^L (\phi(x))^2 dx} > 0.$$

## Nonuniform Rod

1

**Nonuniform Rod:** Assume  $c(x)$ ,  $\rho(x)$ , and  $K_0(x)$  nonconstant:

$$\text{PDE: } c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right), \quad \text{BC: } u(0, t) = 0, \\ \frac{\partial u}{\partial x}(L, t) = 0.$$

$$\text{IC: } u(x, 0) = f(x),$$

*Separation of Variables:*  $u(x, t) = \phi(x)h(t)$  gives:

$$\frac{h'}{h} = \frac{\frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right)}{c\rho\phi} = -\lambda.$$

**Time solution** is  $h(t) = ce^{-\lambda t}$ .

**Sturm-Liouville Problem** is

$$\frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right) + \lambda c\rho\phi = 0, \quad \phi(0) = 0 \quad \text{and} \quad \phi'(L) = 0.$$

## Nonuniform Rod

2

**Theorems** give an infinite sequence of *eigenvalues*,  $\lambda_n$ , and corresponding *eigenfunctions*,  $\phi_n(x)$

Finding  $\phi_n$  might be difficult, but solutions exist

**Superposition principle** gives

$$u(x, t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t},$$

which with **IC** gives

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

## Nonuniform Rod

If  $f(x)$  is *piecewise smooth*, the theorems imply:

$$a_n = \frac{\int_0^L f(x)\phi_n(x)c(x)\rho(x)dx}{\int_0^L \phi_n^2(x)c(x)\rho(x)dx},$$

using the *orthogonality relation*

$$\int_0^L \phi_n(x)\phi_m(x)c(x)\rho(x)dx = 0, \quad n \neq m.$$

For **large time**, the solution takes the shape of the of the first eigenfunction,

$$u(x, t) \approx a_1\phi_1(x)e^{-\lambda_1 t}.$$

## Nonuniform Rod

The **Rayleigh quotient** gives:

$$\lambda = \frac{\int_0^L K_0(x) (\phi'(x))^2 dx}{\int_0^L \phi^2(x) c(x) \rho(x) dx}.$$

The **BC**  $\phi(0) = 0$  implies that a constant *eigenfunction* is not possible.

Since  $(\phi'(x))^2 > 0$ , so the **Rayleigh quotient** implies that  $\lambda > 0$ .

Since all *eigenvalues* are greater than zero, the solution decays to zero.

This is what we expect for a **physical problem** with heat lost on the left end.