Math 531 - Partial Differential Equations
Sturm-Liouville Problems

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Outline

1. Heat Problems
2. Sturm-Liouville Eigenvalue Problem
   - Theorems
   - Nonuniform Rod

Heat in Nonuniform Rod 1

Heat Flow in Nonuniform Rod: Suppose that the ***specific heat***, $c(x)$, ***density***, $\rho(x)$, and ***thermal conductivity***, $K_0(x)$, all depend on the spatial variable x

Suppose that the ***heat source*** $Q(x,t) = \alpha(x)u(x,t)$ satisfies **Newton's Law of Cooling**, which is proportional to heat in the bar (with environmental temperature being zero)

From before, this gives the **Heat Equation**:

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u,$$

which is ***homogeneous***

Heat in Nonuniform Rod 2

Heat Flow in Nonuniform Rod (cont): Apply ***separation of variables***, $u(x,t) = \phi(x)h(t)$, to the **PDE** and rearrange to

$$\frac{h'}{h} = \frac{1}{c\rho\phi}\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \frac{\alpha}{c\rho} = -\lambda.$$

The **differential equation** in x is

$$\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\phi = 0.$$

This is a **Sturm-Liouville Problem**, if there are ***homogeneous BCs***

Solution to this ***differential equation*** may be difficult to find.

Circularly Symmetric Heat Flow 1

Circularly Symmetric Heat Flow: Consider a circularly symmetric region with a uniform material, so $k = \frac{K_0}{c\rho}$, the **Heat Equation** is

$$\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Apply ***separation of variables***, $u(r,t) = \phi(r)h(t)$, to the **PDE** and rearrange to

$$\frac{h'}{kh} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda.$$

The **differential equation** in r is

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0.$$

Circularly Symmetric Heat Flow 2

The **Sturm-Liouville Problem** in r is

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0,$$

if there are ***homogeneous BCs***

For an ***annulus***, the ***homogeneous BCs*** are

$$u(a,t) = 0 \qquad \text{and} \qquad u(b,t) = 0.$$

For a ***circular region***, the ***homogeneous BCs*** are $u(a,t) = 0$ and a ***singularity condition***

$$|u(0,t)| < +\infty.$$

Sturm-Liouville Eigenvalue Problem

The general Sturm-Liouville differential equation:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0,$$

where λ is an eigenvalue, $\quad a < x < b$.

Examples to date are as follows:

1. If $p(x) = \sigma(x) = 1$ and $q(x) = 0$, then
$$\phi'' + \lambda\phi = 0.$$
2. ***Nonuniform heat flow***: $K_0 = p(x)$, $c\rho = \sigma(x)$, and $\alpha = q(x)$,
$$\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\phi = 0.$$
3. ***Circular heat flow***: $p(r) = r$, $\sigma(r) = r$, and $q(r) = 0$,
$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0.$$

Sturm-Liouville Eigenvalue Problem

The Sturm-Liouville eigenvalue problem with eigenvalue λ is:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0,$$

and requires ***homogeneous BCs***

BCs	Heat Eqn	String Eqn	Type
$\phi = 0$	Ends zero Temp	Ends fixed	Dirichlet
$\phi' = 0$	Ends insulated	Ends free	Neumann
$\phi' = \pm h\phi$	Newton's cooling	Elastic boundary	Robin
$\phi(-L) = \phi(L)$ $\phi'(-L) = \phi'(L)$	Perfect thermal contact		Periodic
$\lvert\phi(0)\rvert < \infty$	Bounded Temp		Singularity

Regular Sturm-Liouville Eigenvalue Problem

Consider the ***second order differential equation***:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \qquad a < x < b.$$

The ***homogeneous BCs*** are:

$$\begin{aligned} \beta_1\phi(a) + \beta_2\phi'(a) &= 0, \\ \beta_3\phi(b) + \beta_4\phi'(b) &= 0, \end{aligned}$$

which exclude **periodic** and **singular BCs**.

The following conditions hold:

- β_i are real
- The functions $p(x)$, $q(x)$, and $\sigma(x)$ are continuous and real for $x \in [a, b]$ (including the endpoints)
- $p(x) > 0$ and $\sigma(x) > 0$ for $x \in [a, b]$ (including the endpoints)

Important Theorems

Important Theorems: State and later prove some.

1. All ***eigenvalues*** are **real**.
2. There exist infinitely many ***eigenvalues***, $\lambda_1 < \lambda_2 < ... < \lambda_n < ...$
   - a. There is a smallest ***eigenvalue***, denoted λ_1.
   - b. There is not a largest ***eigenvalue***, *i.e.*, $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each ***eigenvalue***, λ_n, there is an ***eigenfunction***, $\phi_n(x)$, and $\phi_n(x)$ has exactly $n - 1$ zeros for $x \in (a, b)$.
4. The ***eigenfunctions***, $\phi_n(x)$, form a **complete set**, meaning that any ***piecewise smooth*** function $f(x)$ can be represented by a generalized **Fourier series**:
$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x)$$
Furthermore, the infinite series converges to $[f(x^+) + f(x^-)]/2$ for all $x \in (a, b)$ (with appropriate a_n)

Important Theorems

Important Theorems: State and later prove some.

5. ***Eigenfunctions*** corresponding to different ***eigenvalues*** are **orthogonal** relative to the weight function, $\sigma(x)$,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx = 0, \qquad \text{if} \quad \lambda_n \neq \lambda_m.$$
.
6. Any ***eigenvalue*** can be related to its ***eigenfunction*** by the **Rayleigh quotient**
$$\lambda = \frac{-p(x)\phi(x)\phi'(x)\Big|_a^b + \int_a^b \left[p(x)\left(\frac{d\phi}{dx}\right)^2 - q(x)\phi^2(x)\right]dx}{\int_a^b \phi^2(x)\sigma(x)dx},$$
where the **BCs** may simplify this expression.

Example

Example: Consider the ***Sturm-Liouville eigenvalue problem***:

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

We previously found the ***eigenvalues***, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, with ***eigenfunctions***, $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ for $n = 1, 2, ...$

1. Found real eigenvalues, must establish not complex.
2. Smallest eigenvalue is $\lambda_1 = \left(\frac{\pi}{L}\right)^2$, and clearly $\lambda_n \to \infty$ as $n \to \infty$.
3. Easily seen that $\phi_n(x)$ has $n - 1$ zeros for $x \in (0, L)$.
4. Established **Fourier series** for this SL Problem, and showed ***orthogonality*** of $\phi_n(x)$.
5. The ***Rayleigh quotient*** simplifies to
$$\lambda = \frac{\int_0^L (\phi'(x))^2 dx}{\int_0^L (\phi(x))^2 dx} > 0.$$

Nonuniform Rod

Nonuniform Rod: Assume $c(x)$, $\rho(x)$, and $K_0(x)$ nonconstant:

PDE: $c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)$,

BC: $u(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$.

IC: $u(x,0) = f(x)$,

Separation of Variables: $u(x,t) = \phi(x)h(t)$ gives:

$$\frac{h'}{h} = \frac{\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right)}{c\rho\phi} = -\lambda.$$

Time solution is $h(t) = ce^{-\lambda t}$.

Sturm-Liouville Problem is

$$\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \lambda c\rho\phi = 0, \qquad \phi(0) = 0 \quad \text{and} \quad \phi'(L) = 0.$$

Nonuniform Rod

Theorems give an infinite sequence of ***eigenvalues***, λ_n, and corresponding ***eigenfunctions***, $\phi_n(x)$

Finding ϕ_n might be difficult, but solutions exist

Superposition principle gives

$$u(x,t) = \sum_{n=1}^{\infty} a_n\phi_n(x)e^{-\lambda_n t},$$

which with **IC** gives

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x).$$

Nonuniform Rod

If $f(x)$ is ***piecewise smooth***, the theorems imply:

$$a_n = \frac{\int_0^L f(x)\phi_n(x)c(x)\rho(x)dx}{\int_0^L \phi_n^2(x)c(x)\rho(x)dx},$$

using the ***orthogonality relation***

$$\int_0^L \phi_n(x)\phi_m(x)c(x)\rho(x)dx = 0, \qquad n \neq m.$$

For **large time**, the solution takes the shape of the of the first eigenfunction,

$$u(x,t) \approx a_1\phi_1(x)e^{-\lambda_1 t}.$$

Nonuniform Rod

The **Rayleigh quotient** gives:

$$\lambda = \frac{\int_0^L K_0(x)\left(\phi'(x)\right)^2 dx}{\int_0^L \phi^2(x)c(x)\rho(x)dx}.$$

The **BC** $\phi(0) = 0$ implies that a constant ***eigenfunction*** is not possible.

Since $\left(\phi'(x)\right)^2 > 0$, so the ***Rayleigh quotient*** implies that $\lambda > 0$.

Since all ***eigenvalues*** are greater than zero, the solution decays to zero.

This is what we expect for a ***physical problem*** with heat lost on the left end.