Math 531 - Partial Differential Equations

Separation of Variables – Part B

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Outline

Heat Equation - Insulated BCs

The **Heat Equation - Insulated BCs**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad 0 < x < L,$$

with initial conditions (ICs) and **Neumann or Insulated boundary conditions** (BCs):

$$u(x,0) = f(x), \quad 0 < x < L, \quad \text{with} \quad u_x(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0.$$

Separation of Variables: Again we separate the temperature $u(x,t)$ into a product of a function of x and a function of t

$$u(x,t) = \phi(x)G(t)$$

From the **PDE** we have

$$\phi G' = k\phi'' G \qquad \text{or} \qquad \frac{G'}{kG} = \frac{\phi''}{\phi} = -\lambda$$

Heat Equation - Insulated BCs

Two ODEs: The separation of variables leaves to ODEs. The time-varying ODE is:

$$G' = -k\lambda G,$$

which has the solution

$$G(t) = Ae^{-k\lambda t}.$$

The associated **Sturm-Liouville/BVP** in space, x, is

$$\phi'' + \lambda\phi = 0 \qquad \text{with} \quad \phi'(0) = 0 \quad \text{and} \quad \phi'(L) = 0.$$

We must consider **3 cases**, depending on λ.

Case (i): Let $\lambda = 0$, then $\phi'' = 0$ or $\phi(x) = c_2 x + c_1$.

The BCs give $\phi'(0) = \phi'(L) = c_2 = 0$. However, c_1 is arbitrary, so we have an eigenvalue $\lambda_0 = 0$ with associated eigenfunction:

$$\phi_0(x) = 1.$$

Heat Equation - Insulated BCs

Case (ii): Let $\lambda = -\alpha^2 < 0$, then $\phi'' - \alpha^2\phi = 0$, so

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

The BC at $x = 0$ gives $\phi'(0) = c_2\alpha = 0$, so $c_2 = 0$.
Similarly, $\phi'(L) = c_1\alpha \sinh(\alpha l) = 0$, so $c_1 = 0$.
Thus, if $\lambda < 0$, only the ***trivial solution***, $\phi(x) \equiv 0$, satisfies the BCs.

Case (iii): Let $\lambda = \alpha^2 > 0$, then $\phi'' + \alpha^2\phi = 0$, so

$$\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x).$$

The BC at $x = 0$ gives $\phi'(0) = c_2\alpha = 0$, so $c_2 = 0$.
The other BC gives $\phi'(L) = -c_1\alpha \sin(\alpha L) = 0$.
Since we do NOT want the trivial solution, we need $\sin(\alpha L) = 0$ or $\alpha L = n\pi, \quad n = 1, 2, ...$ or

$$\alpha_n = \frac{n\pi}{L} \qquad \text{or} \qquad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, ...$$

Heat Equation - Insulated BCs

Case (iii): (cont.) Since the arbitrary constant is associated with the cosine function, the **eigenfunction** is:

$$\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right).$$

The product solutions are:

$$u_0(x,t) = 1 \qquad \text{and} \qquad u_n(x,t) = e^{-\frac{kn^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right).$$

The **Superposition Principle** gives the solution:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{kn^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right).$$

Orthogonality of Cosines

Assume $m \neq n$, integers and with some trig identities consider

$$\begin{aligned}
\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx &= \int_0^L \frac{\cos\left(\frac{(n-m)\pi x}{L}\right)+\cos\left(\frac{(n+m)\pi x}{L}\right)}{2}dx \\
&= \frac{1}{2}\left(\frac{\sin\left(\frac{(n-m)\pi x}{L}\right)}{(n-m)\pi/L}+\frac{\sin\left(\frac{(n+m)\pi x}{L}\right)}{(n+m)\pi/L}\right)\Bigg|_0^L \\
&= 0
\end{aligned}$$

When $m = n$, then

$$\begin{aligned}
\int_0^L \cos^2\left(\frac{n\pi x}{L}\right)dx &= \int_0^L \frac{1+\cos\left(\frac{2n\pi x}{L}\right)}{2}dx \\
&= \left(\frac{x}{2}+\frac{\sin\left(\frac{2n\pi x}{L}\right)}{4n\pi/L}\right)\Bigg|_0^L \\
&= \frac{L}{2}
\end{aligned}$$

Orthogonality of $\phi_0(x)$ and $\phi_n(x)$

Consider $\phi_0(x) = 1$ and $\phi_n(x)$, and integrate

$$\int_0^L 1 \cdot \cos\left(\frac{n\pi x}{L}\right) dx = \frac{L}{n\pi}\left(\sin\left(\frac{n\pi x}{L}\right)\right)\Big|_0^L = 0.$$

Also,

$$\int_0^L (1 \cdot 1) dx = L.$$

The **eigenfunctions**, $\phi_i(x)$, $\quad i = 0, 1, 2, ...$, are mutually ***orthogonal***, which allows finding **Fourier coefficients** for any **initial conditions**, $f(x)$, where

$$u(x, 0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$$

Fourier Coefficients

For

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

we first multiply by $\phi_0(x) = 1$ and integrate $x \in [0, L]$, which by orthogonality with $\phi_n(x), n = 1, 2, \dots$ gives

$$\int_0^L f(x)dx = \int_0^L A_0 dx = A_0 L, \qquad \text{or} \qquad A_0 = \frac{1}{L}\int_0^L f(x)dx.$$

Next we multiply by $\phi_m(x)$ and integrate $x \in [0, L]$, so

$$\begin{aligned}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx &= \sum_{n=1}^{\infty} A_n \int_0^L \left(\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right)dx, \\ &= A_m\left(\frac{L}{2}\right)\end{aligned}$$

from orthogonality.

Fourier Coefficients

It follows that the **Fourier coefficients** are:

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \qquad \text{and} \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

Recall the solution of the **heat equation** with **insulated boundaries conditions** is given by:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{kn^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right).$$

The **steady-state solution** examines $t \to \infty$,

$$\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)dx,$$

which is the ***average temperature distribution*** from the **ICs**.

Heat Conduction in a Ring 1

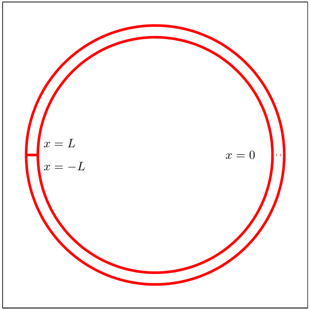
Heat Conduction in a Ring: Here we consider a thin, insulated wire that is deformed into a ring.

The model satisfies the **heat equation**.

PDE: $u_t = ku_{xx}$,
$t > 0, \quad -L < x < L$,

BC: Periodic (homogeneous):
$u(-L,t) = u(L,t)$,
$u_x(-L,t) = u_x(L,t)$,

IC: $u(x,0) = f(x), \quad -L < x < L$.



Heat Conduction in a Ring 2

The PDE for the **Heat Equation in a Ring** separates as before, so if $u(x,t) = \phi(x)G(t)$, then

$$\phi G' = k\phi'' G \qquad \text{or} \qquad \frac{G'}{kG} = \frac{\phi''}{\phi} = -\lambda$$

Again the ***time-varying ODE*** is:

$$G' = -k\lambda G,$$

which has the solution

$$G(t) = Ae^{-k\lambda t}.$$

Heat Conduction in a Ring 3

The associated **Sturm-Liouville/BVP** in space, x, is

$$\phi'' + \lambda\phi = 0 \qquad \text{with} \quad \phi(-L) = \phi(L) \quad \text{and} \quad \phi'(-L) = \phi'(L).$$

Case (i): Let $\lambda = 0$, then $\phi'' = 0$ or $\phi(x) = c_2 x + c_1$.

The BCs give $\phi(-L) - \phi(L) = -2c_2 L = 0$ or $c_2 = 0$.

Also, $\phi'(-L) - \phi'(L) = c_2 - c_2 = 0$, which gives no new information.

Thus, c_1 is arbitrary, so we have an eigenvalue $\lambda_0 = 0$ with associated eigenfunction:

$$\phi_0(x) = 1.$$

Heat Conduction in a Ring 4

Case (ii): Let $\lambda = -\alpha^2 < 0$, then $\phi'' - \alpha^2\phi = 0$, so

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

The first BC gives
$c_1 \cosh(-\alpha L) + c_2 \sinh(-\alpha L) = c_1 \cosh(\alpha L) + c_2 \sinh(\alpha L)$, so
$2c_2 \sinh(\alpha L) = 0$ (from cosh being even and sinh being odd). Hence, $c_2 = 0$.

The second BC gives
$c_1\alpha \sinh(-\alpha L) + c_2\alpha \cosh(-\alpha L) = c_1\alpha \sinh(\alpha L) + c_2\alpha \cosh(\alpha L)$, so
$2c_1\alpha \sinh(\alpha L) = 0$ or $c_1 = 0$.

Thus, if $\lambda < 0$, only the ***trivial solution***, $\phi(x) \equiv 0$, satisfies the BCs.

Heat Conduction in a Ring 5

Case (iii): Let $\lambda = \alpha^2 > 0$, then $\phi'' + \alpha^2\phi = 0$, so

$$\phi(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x).$$

The first BC gives
$c_1\cos(-\alpha L) + c_2\sin(-\alpha L) = c_1\cos(\alpha L) + c_2\sin(\alpha L)$, so
$2c_2\sin(\alpha L) = 0$ (from cos being even and sin being odd), which has nontrivial solutions, $c_2 \neq 0$, when $\alpha_n = n\pi/L$, $n = 1, 2, ...$

The second BC gives
$-c_1\alpha\sin(-\alpha L) + c_2\alpha\cos(-\alpha L) = -c_1\alpha\sin(\alpha L) + c_2\alpha\cos(\alpha L)$, so
$2c_1\alpha\sin(\alpha L) = 0$, which has nontrivial solutions, $c_1 \neq 0$, when $\alpha_n = n\pi/L$, $n = 1, 2, ...$

It follows that $\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, ...$, are **eigenvalues** with corresponding independent **eigenfunctions**

$$\phi_n(x) = A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, ...$$

Heat Conduction in a Ring 6

The product solutions are:

$$
\begin{aligned}
u_0(x,t) &= A_0 \\
u_n(x,t) &= e^{-\frac{kn^2\pi^2 t}{L^2}} \left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right).
\end{aligned}
$$

The **Superposition Principle** gives the solution:

$$
u(x,t) = A_0 + \sum_{n=1}^{\infty} e^{-\frac{kn^2\pi^2 t}{L^2}} \left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right).
$$

The **Initial Condition** gives

$$
u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right).
$$

Orthogonality

The **orthogonality** over $x \in (-L, L)$ give

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0. \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0. \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0 \quad \text{for all} \quad n > 0, \quad m \geq 0.$$

The **Fourier coefficients** are

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

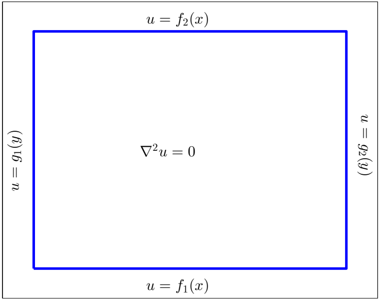
Laplace's Equation 1

Laplace's Equation on a Rectangle: Consider a rectangular region, $0 \le x \le L$ and $0 \le y \le H$. We seek the steady-state temperature distribution in this rectangle

Laplace's Equation satisfies:

PDE: $\nabla^2 u = 0$,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

BC's: $u(x,0) = f_1(x)$,
$u(x,H) = f_2(x)$,
$u(0,y) = g_1(y)$,
$u(L,y) = g_2(y)$



This problem has 4 nonhomogeneous BC's

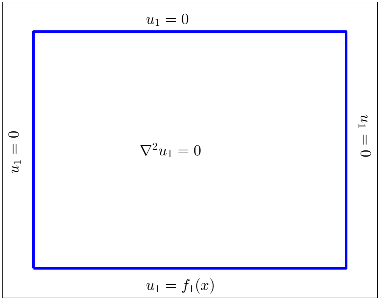
Laplace's Equation 2

Laplace's Equation on a Rectangle: It is easier to use the ***superposition principle*** and divide the problem into 4 problems, each with only **one** nonhomogeneous BC

Laplace's Equation satisfies:

PDE: $\nabla^2 u_1 = 0$,
$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0.$$

BC's: $u_1(x,0) = f_1(x)$,
$u_1(x,H) = 0$,
$u_1(0,y) = 0$,
$u_1(L,y) = 0$



This problem is readily solved with our Separation of Variables technique. (Similarly, for the other 3 problems.)

Laplace's Equation

Laplace's Equation on a Rectangle: Consider the problem:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \qquad 0 < x < L \quad \text{and} \quad 0 < y < H.$$

The **BCs** are

$$\begin{array}{cc} u_1(x,0) = f_1(x), & u_1(x,H) = 0, \\ u_1(0,y) = 0, & u_1(L,y) = 0. \end{array}$$

Assume $u(x,y) = \phi(x)\psi(y)$, then the PDE becomes

$$\phi''\psi + \phi\psi'' = 0 \qquad \text{or} \qquad \frac{\phi''(x)}{\phi(x)} = -\frac{\psi''(y)}{\psi(y)} = -\lambda,$$

which is a constant because each side of the equation varies independently in either x or y.

Laplace's Equation 4

From our separation assumption the **homogeneous BCs** imply that

$$\psi(H) = 0, \qquad \phi(0) = 0, \qquad \text{and} \qquad \phi(L) = 0.$$

We need to locate our **Sturm-Liouville problem** to obtain our **eigenvalues** and **eigenfunctions** for this PDE.

Significantly, we find the pairwise homogeneous BC conditions, which in this case are associated with $\phi(x)$, so examine

$$\phi'' + \lambda\phi = 0, \qquad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

This **eigenvalue problem** is familiar from before with

Eigenvalues: $\lambda_n = \dfrac{n^2\pi^2}{L^2}, \quad n = 1, 2, \dots$

Eigenfunctions: $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \dots$

Laplace's Equation 5

With the **eigenvalues**, $\lambda_n = \frac{n^2\pi^2}{L^2}$, we solve the second ODE:

$$\psi'' - \frac{n^2\pi^2}{L^2}\psi = 0, \qquad \text{with} \quad \psi(H) = 0.$$

With the homogeneous boundary condition, it suggests selecting the ***linearly independent*** solutions:

$$\psi(y) = c_1 \cosh\left(\frac{n\pi(H-y)}{L}\right) + c_2 \sinh\left(\frac{n\pi(H-y)}{L}\right).$$

The BC, $\psi(H) = 0$, gives $\quad \psi(H) = c_1 = 0$, so

$$\psi_n(y) = c_2 \sinh\left(\frac{n\pi(H-y)}{L}\right).$$

The results above are combined with $u_n(x,y) = \phi_n(x)\psi_n(x)$

Laplace's Equation 6

The **extended superposition principle** gives the following solution:

$$u_1(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(H-y)}{L}\right).$$

It remains to examine the **nonhomogeneous BC**

$$u_1(x,0) = f_1(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi H}{L}\right)$$

We use the **orthogonality of the sines** to obtain the **Fourier coefficients**

$$B_n \sinh\left(\frac{n\pi H}{L}\right) = \frac{2}{L}\int_0^L f_1(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Laplace's Equation 7

This process could be repeated for each of the other Dirichlet BCs to find the **3** other solutions with ***3 homogeneous BCs***

For example, if $u_2(0,y) = g_1(y)$ (other BCs homogeneous), then the same procedure above gives

$$u_2(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi(L-x)}{H}\right)\sin\left(\frac{n\pi y}{H}\right),$$

where the **Fourier coefficient** satisfies

$$C_n = \frac{2}{H\sinh\left(\frac{n\pi L}{H}\right)}\int_0^H g_1(x)\sin\left(\frac{n\pi y}{H}\right)dy.$$

We solve all these problems, then the general **Laplace's equation** satisfies

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

Laplace's Equation 1

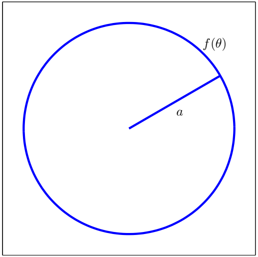
Laplace's Equation - Circular Disk: Consider a circular region, $0 \le r \le a$ and $-\pi < \theta \le \pi$. Find the steady-state temperature distribution.

Laplace's Equation satisfies:

PDE: $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$

BC: $u(a, \theta) = f(\theta)$,

This problem has **periodic BCs** (homogeneous):



$$u(r, -\pi) = u(r, \pi) \qquad \text{and} \qquad u_\theta(r, -\pi) = u_\theta(r, \pi).$$

There is an **implicit BC** that solutions are bounded, so

$$|u(0, \theta)| < \infty.$$

Laplace's Equation 2

Separation of Variables: Let $u(r,\theta) = \phi(\theta)G(r)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi + \frac{1}{r^2}G\phi'' = 0.$$

This gives

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{\phi''}{\phi} = \lambda.$$

The **Sturm-Liouville problem** has the **eigenvalue problem**:

$$\phi'' + \lambda\phi = 0,$$

where the ***periodic BCs*** on $u(r,\theta)$ imply that

$$\phi(-\pi) = \phi(\pi) \qquad \text{and} \qquad \phi'(-\pi) = \phi'(\pi).$$

Laplace's Equation 4

Earlier we saw that the **Sturm-Liouville problem**:

$$\phi'' + \lambda\phi = 0, \qquad \phi(-\pi) = \phi(\pi) \quad \text{and} \quad \phi'(-\pi) = \phi'(\pi),$$

with periodic BCs satisfies the following:

1. If $\lambda < 0$, then only the ***trivial solution*** exists.
2. For $\lambda_0 = 0$, there is the ***eigenfunction***

$$\phi_0(x) = 1.$$

3. For $\lambda = \alpha^2 > 0$, we obtain ***eigenvalues*** and ***eigenfunctions***:

$$\lambda_n = n^2, \qquad \phi_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \qquad n = 1, 2, ...$$

Laplace's Equation

From the separation of variables, the r equation becomes

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2G$$

or

$$r^2G'' + rG' - n^2G = 0.$$

When $n = 0$, the r equation satisfies:

$$\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0,$$

which is integrated twice to give

$$\begin{aligned} r\frac{dG}{dr} &= c_1, \\ G(r) &= c_1\ln(r) + c_2. \end{aligned}$$

Laplace's Equation

Thus, for $\lambda_0 = 0$, we have $G_0(r) = c_1 \ln(r) + c_2$.

The **boundedness BC** as $r \to 0$ implies $c_1 = 0$, so

$$G_0(r) = c_2$$

For $n > 0$, the differential equation in $G(r)$ is **Euler's equation**:

$$r^2 G'' + rG - n^2 G = 0,$$

which is solved by using $G(r) = cr^\alpha$, so $G'(r) = c\alpha r^{\alpha-1}$ and $G''(r) = c\alpha(\alpha-1)r^{\alpha-2}$ or

$$\begin{aligned} c\alpha(\alpha-1)r^\alpha + c\alpha r^\alpha - n^2 c r^\alpha &= 0, \\ cr^\alpha(\alpha^2 - n^2) &= 0 \end{aligned}$$

Thus, the general solution to this **Euler's equation** is:

$$G_n(r) = c_1 r^{-n} + c_2 r^n.$$

Laplace's Equation

The **boundedness BC** as $r \to 0$ implies for $G_n(r) = c_1 r^{-n} + c_2 r^n$ that $c_1 = 0$, so

$$G_n(r) = c_2 r^n.$$

Combining the results above with the **Superposition Principle** gives:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta),$$
$$0 \le r < a, \quad -\pi < \theta \le \pi.$$

Laplace's Equation 8

Applying the BC at $r = a$ gives:

$$u(a,\theta) = f(\theta) = A_0 + \sum_{n=1}^{\infty} A_n a^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n a^n \sin(n\theta), \quad -\pi < \theta \le \pi.$$

From the **orthogonality**, the **Fourier coefficients** are

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

$$A_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)d\theta \qquad B_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)d\theta$$

Note that in **Steady-state** the temperature at the center of the disk is the average of the perimeter temperature

Mean Value Theorem

Theorem (Mean Value Theorem)

The average solution of Laplace's equation inside a circle gives the temperature at the center (origin or $r = 0$),

$$u(0, \theta) = A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta.$$

The temperature at the center of a circle is the average of the temperature around any circle of radius, r_0, (inside R)

Maximum Principle

Theorem (Maximum Principle)

*In steady state, the temperature cannot attain its **maximum** (or **minimum**) in the interior unless the temperature is constant everywhere (assuming no sources or sinks).*

Sketch of Proof: Assume there is a maximum at a point P inside R. Create a small circle about P completely inside R. If it is the maximum point, then it can only be the average of the surrounding circle if all points on the circle are also maximum points. Thus, all points throughout the region have the same value.

It follows that the **maximum** and **minimum** temperatures occur on the boundary of R.

Well-posedness

A problem is **well-posed** if there exists a unique solution that depends continuously on the nonhomogeneous data, *i.e.*, small variations in the data result in small changes in the solution

Consider

$$\nabla^2 u = 0 \quad \text{on} \quad R \qquad \text{with} \quad u = f(x) \quad \text{on} \quad \partial R.$$

Consider a small variation on the boundary, ∂R, with $g(x) \approx f(x)$

$$\nabla^2 v = 0 \quad \text{on} \quad R \qquad \text{with} \quad v = g(x) \quad \text{on} \quad \partial R.$$

Let $w = u - v$. Clearly,

$$\nabla^2 w = 0 \quad \text{on} \quad R \qquad \text{with} \quad w = f(x) - g(x) \quad \text{on} \quad \partial R.$$

Well-posedness

Since

$$\nabla^2 w = 0 \quad \text{on} \quad R \qquad \text{with} \quad w = f(x) - g(x) \quad \text{on} \quad \partial R,$$

the **Maximum (and minimum) principle** give the maximum and minimum of the solution occur on the boundary, ∂R.

It follows that

$$\min(f(x) - g(x)) \le w \le \max(f(x) - g(x)) \quad \text{for all } x \in R.$$

Thus, if $f(x)$ is close to $g(x)$, then w is small everywhere in R

Uniqueness

Theorem (Uniqueness)

If $u(x)$ is a solution of

$$\nabla^2 u = 0 \quad \text{for} \quad x \in R \qquad \text{with} \quad u = f(x) \quad \text{on} \quad \partial R,$$

*then $u(x)$ is **unique**.*

Proof: Suppose there is another $v(x)$ with $\nabla^2 v = 0$ and $v = f(x)$ on ∂R. Let $w = u - v$, then

$$\nabla^2 w = 0 \quad \text{for} \quad x \in R \qquad \text{with} \quad w = 0 \quad \text{on} \quad \partial R.$$

The **Maximum principle** implies $w(x) \equiv 0$. Thus, $u(x) = v(x)$, so $u(x)$ is **unique**. Q.E.D.

Solvability Condition

If the **heat flow** is specified, $-K_0\nabla u \cdot \tilde{\mathbf{n}}$, on the boundary, ∂R and suppose that

$$\nabla^2 u = 0 \quad \text{on} R.$$

According to the **Divergence Theorem**, we have

$$\iint_R \nabla^2 u \, dA = \iint_R \nabla \cdot \nabla u \, dA = \oint_{\partial R} \nabla u \cdot \tilde{\mathbf{n}} \, dS.$$

Thus, when u satisfies **Laplace's equation**, then the net ***heat flow*** through the boundary, ∂R, must be **zero** for the **solvability (compatibility) condition**.