

Math 531 - Partial Differential Equations

PDEs - Higher Dimensions

Part A

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 - Green's Formula and Self-Adjoint
 - Orthogonality
 - Gram-Schmidt Process

Introduction

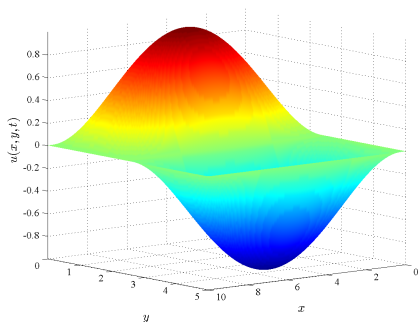
We want to consider **PDEs** in higher dimensions.

Vibrating Membrane:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Heat Conduction:

$$\frac{\partial u}{\partial t} = k \nabla^2 u$$



Rectangular Membrane

Vibrating Rectangular Membrane:

PDE:

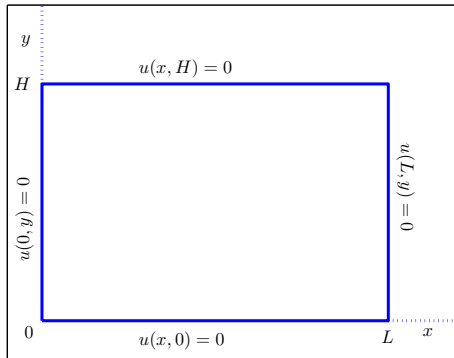
$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

BCs:

$$\begin{aligned} u(x, 0, t) &= 0, \\ u(x, H, t) &= 0, \\ u(0, y, t) &= 0, \\ u(L, y, t) &= 0, \end{aligned}$$

ICs:

$$u(x, y, 0) = \alpha(x, y) \quad \text{and} \quad u_t(x, y, 0) = \beta(x, y).$$



Rectangular Membrane

Let $u(x, y, t) = h(t)\phi(x)\psi(y)$, then the **PDE** becomes

$$h''\phi\psi = c^2 (h\phi''\psi + h\phi\psi'').$$

This is rearranged to give

$$\frac{h''}{c^2 h} = \frac{\phi''}{\phi} + \frac{\psi''}{\psi} = -\lambda,$$

which gives the time dependent ODE:

$$h'' + \lambda c^2 h = 0.$$

The remaining *spatial equation* is rearranged to:

$$\frac{\phi''}{\phi} = -\frac{\psi''}{\psi} - \lambda = -\mu.$$

Rectangular Membrane

The *spatial equations* form two *Sturm-Liouville problems*. With the **BCs** $u(0, y) = 0 = u(L, y)$, we obtain the **1st Sturm-Liouville problem**:

$$\phi'' + \mu\phi = 0, \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

From before, this gives the *eigenvalues* and *eigenfunctions*:

$$\mu_m = \frac{m^2\pi^2}{L^2} \quad \text{and} \quad \phi_m(x) = \sin\left(\frac{m\pi x}{L}\right).$$

If $\lambda - \mu_m = \nu$, then the **2nd Sturm-Liouville problem** is:

$$\psi'' + \nu\psi = 0, \quad \psi(0) = 0 \quad \text{and} \quad \psi(H) = 0.$$

From before, this gives the *eigenvalues* and *eigenfunctions*:

$$\nu_n = \frac{n^2\pi^2}{H^2} \quad \text{and} \quad \psi_n(y) = \sin\left(\frac{n\pi y}{H}\right).$$

Rectangular Membrane

From above we see $\lambda_{mn} = \mu_m + \nu_n = \frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{H^2} > 0$, so the time equation:

$$h'' + \lambda c^2 h = 0,$$

has the solution

$$h_{mn}(t) = a_n \cos(c\sqrt{\lambda_{mn}}t) + b_n \sin(c\sqrt{\lambda_{mn}}t).$$

The **Product solution** is

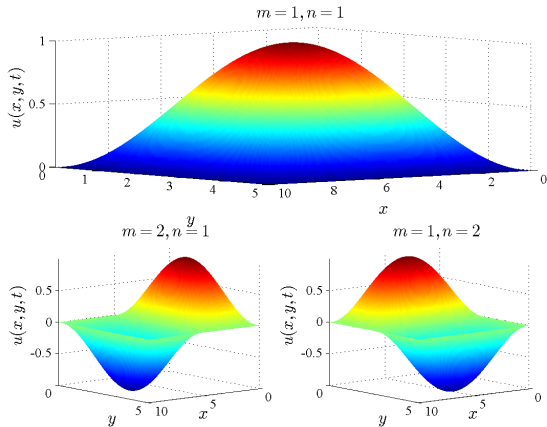
$$u_{mn}(t) = \left(a_{mn} \cos\left(c\sqrt{\lambda_{mn}}t\right) + b_{mn} \sin\left(c\sqrt{\lambda_{mn}}t\right) \right) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

The **Superposition Principle** gives

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(a_{mn} \cos\left(c\sqrt{\lambda_{mn}}t\right) + b_{mn} \sin\left(c\sqrt{\lambda_{mn}}t\right) \right) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

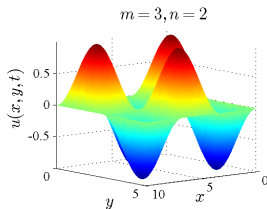
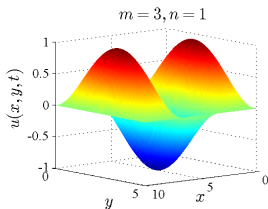
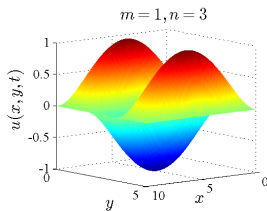
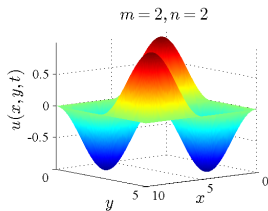
Nodal Curves

Nodal Curves



Nodal Curves

Nodal Curves



Rectangular Membrane

From the **ICs**, we have

$$u(x, y, 0) = \alpha(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

Multiply by $\sin\left(\frac{j\pi x}{L}\right)$ and integrate $x \in [0, L]$ and $\sin\left(\frac{n\pi y}{H}\right)$ and integrate $y \in [0, H]$. **Orthogonality** gives:

$$a_{mn} = \frac{4}{LH} \int_0^H \int_0^L \alpha(x, y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) dx dy.$$

Similarly,

$$u_t(x, y, 0) = \beta(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} c\sqrt{\lambda_{mn}} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right),$$

and **orthogonality** gives:

$$b_{mn} = \frac{4}{LHc\sqrt{\lambda_{mn}}} \int_0^H \int_0^L \beta(x, y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) dx dy.$$

Theorems for Eigenvalue Problems

Helmholtz Equation:

$$\nabla^2 \phi + \lambda \phi = 0 \quad \text{in } R,$$

with

$$\alpha \phi + \beta \nabla \phi \cdot \tilde{\mathbf{n}} = 0 \quad \text{on } \partial R.$$

Generalizes to

$$\nabla \cdot (p \nabla \phi) + q \phi + \lambda \sigma \phi = 0.$$

Theorem

1. All *eigenvalues* are real.
2. There exists infinitely many *eigenvalues* with a smallest, but no largest *eigenvalue*.
3. There may be many *eigenfunctions* corresponding to an *eigenvalue*.

Theorems for Eigenvalue Problems

Theorem

4. The **eigenfunctions** form a complete set, so if $f(x, y)$ is **piecewise smooth**

$$f(x, y) \sim \sum_{\lambda} a_{\lambda} \phi_{\lambda}(x, y).$$

5. **Eigenfunctions** corresponding to different **eigenvalues** are **orthogonal**

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} \sigma dR = 0 \quad \text{if } \lambda_1 \neq \lambda_2.$$

Different **eigenfunctions** belonging to the same **eigenvalue** can be made **orthogonal** by **Gram-Schmidt process**.

Theorems for Eigenvalue Problems

Theorem

6. For $\sigma = 1$, an **eigenvalue** λ can be related to the **eigenfunction** by the Rayleigh quotient:

$$\lambda = \frac{-\oint \phi \nabla \phi \cdot n \, ds + \iint_R |\nabla \phi|^2 dR}{\iint_R \phi^2 dR}.$$

The boundary conditions often simplify the boundary integral.

We use the **Example** for the **vibrating rectangular membrane** to illustrate a number of the Theorem results above.

Example

Example: The Sturm-Liouville problem for the *vibrating rectangular membrane* satisfies:

PDE: $\nabla^2 \phi + \lambda \phi = 0,$

ICs: $\phi(0, y) = 0, \quad \phi(L, y) = 0,$

$\phi(x, 0) = 0, \quad \phi(x, H) = 0.$

We have already shown that this **Helmholtz equation** has *eigenvalues*:

$$\lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2, \quad m = 1, 2, \dots \quad n = 1, 2, \dots$$

with corresponding *eigenfunctions*:

$$\phi_{mn}(x, y) = \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right), \quad m = 1, 2, \dots \quad n = 1, 2, \dots$$

Example

Example (cont): We already demonstrated that:

- 1 **Real eigenvalues:** The *eigenvalues* are clearly real.
- 2 **Ordering the eigenvalues:** It is easy to see that there is the lowest *eigenvalue* $\lambda_1 = \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2$ and that there is no largest *eigenvalue*, as m or $n \rightarrow \infty$.
- 3 **Multiple eigenvalues:** Suppose that $L = 2H$. It follows that

$$\lambda_{mn} = \frac{\pi^2}{4H^2} (m^2 + 4n^2).$$

It is easy to see for $m = 4, n = 1$ and $m = 2, n = 2$,

$$\lambda_{41} = \lambda_{22} = \frac{5\pi^2}{H^2}.$$

These solutions will oscillate with the same frequency.

Example

Example (cont): We have:

- ④ **Series of eigenfunctions:** If $f(x, y)$ is *piecewise smooth*, then

$$f(x, y) \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

- ⑤ **Convergence:** As before, write the **Error** using a finite series

$$E = \iint_R \left(f - \sum_{\lambda} a_{\lambda} \phi_{\lambda} \right)^2 dR.$$

The approximation improves with increasing λ , and we found that the series $\sum_{\lambda} a_{\lambda} \phi_{\lambda}$ *converges in the mean* to f .

Orthogonality

Orthogonality: Assume $\lambda_1 \neq \lambda_2$ with *eigenfunctions* ϕ_{λ_1} and ϕ_{λ_2} and insert these into the equation:

$$\nabla \cdot (p \nabla \phi) + q \phi + \lambda \sigma \phi = 0.$$

Multiplying by the other eigenfunction and subtracting, we can write

$$\phi_{\lambda_1} (\nabla \cdot (p \nabla \phi_{\lambda_2})) - \phi_{\lambda_2} (\nabla \cdot (p \nabla \phi_{\lambda_1})) = (\lambda_2 - \lambda_1) \sigma \phi_{\lambda_1} \phi_{\lambda_2}.$$

Use integration by parts over the entire region R and the homogeneous boundary conditions to give (more details next section):

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} \sigma dR = 0, \quad \text{if } \lambda_1 \neq \lambda_2.$$

Fourier Coefficients

Fourier Coefficients: Assume that f is *piecewise smooth*, so

$$f(x, y) \sim \sum_{\lambda} a_{\lambda} \phi_{\lambda}.$$

Use the *orthogonality relationship* with respect to the weighting function σ :

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} \sigma dR = 0, \quad \text{if } \lambda_1 \neq \lambda_2,$$

then the **Fourier coefficients** satisfy

$$a_{\lambda_i} = \frac{\iint_R f \phi_{\lambda_i} \sigma dR}{\iint_R \phi_{\lambda_i}^2 \sigma dR}.$$

Note: If there is more than one *eigenfunction* associated with an *eigenvalue*, then assume the *eigenfunctions* have been made *orthogonal* by *Gram-Schmidt*.

Green's Formula

Consider the **PDE**:

$$\nabla^2 \phi + \lambda \phi = 0, \quad \text{in } R,$$

with **BCs**:

$$\beta_1 \phi + \beta_2 \nabla \phi \cdot \tilde{\mathbf{n}} = 0, \quad \text{on } \partial R,$$

where β_1 and β_2 are real functions in R .

Basic product rule gives:

$$\nabla \cdot (u \nabla v) = u \nabla^2 v + \nabla u \cdot \nabla v,$$

$$\nabla \cdot (v \nabla u) = v \nabla^2 u + \nabla v \cdot \nabla u.$$

Subtracting gives:

$$u \nabla^2 v - v \nabla^2 u = \nabla \cdot (u \nabla v - v \nabla u).$$

Green's Formula

The previous result is integrated to give:

$$\iint_R (u \nabla^2 v - v \nabla^2 u) dR = \iint_R \nabla \cdot (u \nabla v - v \nabla u) dR.$$

Apply the **Divergence Theorem** and obtain:

Green's Formula: Also, **Green's second identity:**

$$\iint_R (u \nabla^2 v - v \nabla^2 u) dR = \oint_{\partial R} (u \nabla v - v \nabla u) \cdot \tilde{\mathbf{n}} dS.$$

This identity is important in showing an operator is **self-adjoint** if there are **homogeneous BCs**.

Self-Adjoint Operator

Let $L = \nabla^2$ be a linear operator:

Theorem (Self-Adjoint)

If u and v are two functions such that

$$\oint_{\partial R} (u \nabla v - v \nabla u) \cdot \tilde{\mathbf{n}} \, dS = 0,$$

then

$$\iint_R (u \nabla^2 v - v \nabla^2 u) \, dR = \iint_R (u L[v] - v L[u]) \, dR = 0.$$

Note: The above theorem is stated in 2D, but it equally applies to 3D by substituting double integrals with triple integrals and line integrals with surface integrals.

Orthogonality

Orthogonality of Eigenfunctions: We use **Green's formula** to show *orthogonality* of *eigenfunctions*, ϕ_1 and ϕ_2 , corresponding to different *eigenvalues*, λ_1 and λ_2 .

Suppose with $L = \nabla^2$

$$L[\phi_1] + \lambda_1\phi_1 = 0 \quad \text{and} \quad L[\phi_2] + \lambda_2\phi_2 = 0.$$

If ϕ_1 and ϕ_2 satisfy the same *homogeneous BCs*,

$$\oint_{\partial R} (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot \tilde{\mathbf{n}} \, dS = 0,$$

then by **Green's formula**:

$$\iint_R (\phi_1 L[\phi_2] - \phi_2 L[\phi_1]) \, dR = 0.$$

Orthogonality

However,

$$\begin{aligned}\iint_R (\phi_1 L[\phi_2] - \phi_2 L[\phi_1]) dR &= \iint_R (\lambda_2 \phi_1 \phi_2 - \lambda_1 \phi_1 \phi_2) dR \\ &= (\lambda_2 - \lambda_1) \iint_R \phi_1 \phi_2 dR = 0.\end{aligned}$$

So for $\lambda_2 \neq \lambda_1$, the *eigenfunctions* are **orthogonal**:

$$\iint_R \phi_1 \phi_2 dR = 0.$$

Gram-Schmidt Process

Gram-Schmidt Process: Suppose that $\phi_1, \phi_2, \dots, \phi_m$, are independent *eigenfunctions* all corresponding to the *eigenvalue*, λ (a **single e.v.**).

Let $\psi_1 = \phi_1$ be an *eigenfunction*.

Any linear combination of *eigenfunctions* is also an *eigenfunction*, so take

$$\psi_2 = \phi_2 + c\psi_1.$$

We want

$$\iint_R \psi_1 \psi_2 dR = 0 = \iint_R \psi_1 (\phi_2 + c\psi_1) dR,$$

so choose

$$c = -\frac{\iint_R \phi_2 \psi_1 dR}{\iint_R \psi_1^2 dR}.$$

Gram-Schmidt Process

Gram-Schmidt Process: Continuing take

$$\psi_3 = \phi_3 + c_1\psi_1 + c_2\psi_2.$$

We want

$$\iint_R \psi_3 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} dR = 0,$$

$$\iint_R (\phi_3 + c_1\psi_1 + c_2\psi_2) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} dR = 0.$$

It follows that

$$c_1 = -\frac{\iint_R \phi_3 \psi_1 dR}{\iint_R \psi_1^2 dR} \quad \text{and} \quad c_2 = -\frac{\iint_R \phi_3 \psi_2 dR}{\iint_R \psi_2^2 dR}.$$

Gram-Schmidt Process

Gram-Schmidt Process: In general,

$$\psi_j = \phi_j - \sum_{i=1}^{j-1} \frac{\iint_R \phi_j \psi_i dR}{\iint_R \psi_i^2 dR} \psi_i.$$

Thus, we can always obtain an *orthogonal set of eigenfunctions*.