

# Math 531 - Partial Differential Equations

## PDEs - Higher Dimensions

### Part A

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Spring 2023



## Introduction

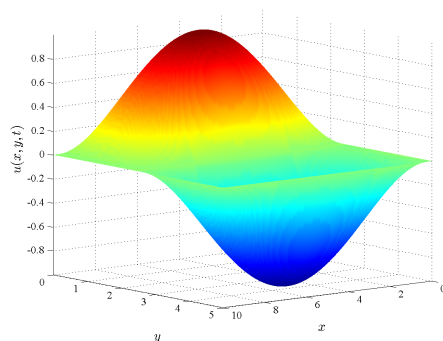
We want to consider **PDEs** in higher dimensions.

**Vibrating Membrane:**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

**Heat Conduction:**

$$\frac{\partial u}{\partial t} = k \nabla^2 u$$



## Outline

- 1 **Vibrating Rectangular Membrane**
  - Separation of Variables
  - Product Solution
  - Fourier Coefficients
- 2 **Theorems for Eigenvalue Problems**
  - Example
  - Orthogonality and Fourier Coefficients
- 3 **More on Multidimensional E.V. Problem**
  - Green's Formula and Self-Adjoint
  - Orthogonality
  - Gram-Schmidt Process



## Rectangular Membrane

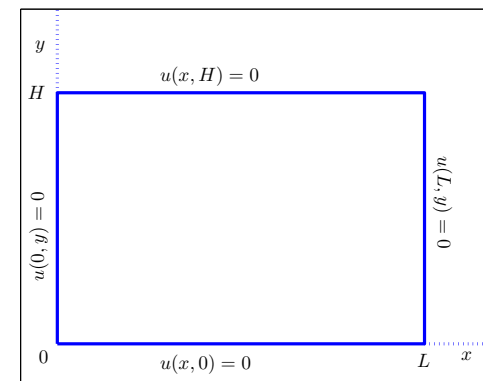
**Vibrating Rectangular Membrane:**

**PDE:**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

**BCs:**

$$\begin{aligned} u(x, 0, t) &= 0, \\ u(x, H, t) &= 0, \\ u(0, y, t) &= 0, \\ u(L, y, t) &= 0, \end{aligned}$$



**ICs:**

$$u(x, y, 0) = \alpha(x, y) \quad \text{and} \quad u_t(x, y, 0) = \beta(x, y).$$



## Rectangular Membrane

Let  $u(x, y, t) = h(t)\phi(x)\psi(y)$ , then the **PDE** becomes

$$h''\phi\psi = c^2(h\phi''\psi + h\phi\psi'')$$

This is rearranged to give

$$\frac{h''}{c^2h} = \frac{\phi''}{\phi} + \frac{\psi''}{\psi} = -\lambda,$$

which gives the time dependent ODE:

$$h'' + \lambda c^2 h = 0.$$

The remaining **spatial equation** is rearranged to:

$$\frac{\phi''}{\phi} = -\frac{\psi''}{\psi} - \lambda = -\mu.$$



## Rectangular Membrane

The **spatial equations** form two **Sturm-Liouville problems**. With the **BCs**  $u(0, y) = 0 = u(L, y)$ , we obtain the 1<sup>st</sup> **Sturm-Liouville problem**:

$$\phi'' + \mu\phi = 0, \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

From before, this gives the **eigenvalues** and **eigenfunctions**:

$$\mu_m = \frac{m^2\pi^2}{L^2} \quad \text{and} \quad \phi_m(x) = \sin\left(\frac{m\pi x}{L}\right).$$

If  $\lambda - \mu_m = \nu$ , then the 2<sup>nd</sup> **Sturm-Liouville problem** is:

$$\psi'' + \nu\psi = 0, \quad \psi(0) = 0 \quad \text{and} \quad \psi(H) = 0.$$

From before, this gives the **eigenvalues** and **eigenfunctions**:

$$\nu_n = \frac{n^2\pi^2}{H^2} \quad \text{and} \quad \psi_n(y) = \sin\left(\frac{n\pi y}{H}\right).$$



## Rectangular Membrane

From above we see  $\lambda_{mn} = \mu_m + \nu_n = \frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{H^2} > 0$ , so the time equation:

$$h'' + \lambda c^2 h = 0,$$

has the solution

$$h_{mn}(t) = a_n \cos(c\sqrt{\lambda_{mn}}t) + b_n \sin(c\sqrt{\lambda_{mn}}t).$$

The **Product solution** is

$$u_{mn}(t) = \left( a_{mn} \cos\left(c\sqrt{\lambda_{mn}}t\right) + b_{mn} \sin\left(c\sqrt{\lambda_{mn}}t\right) \right) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

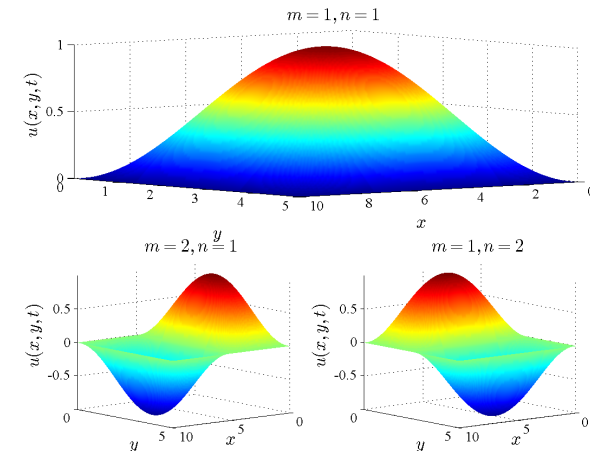
The **Superposition Principle** gives

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( a_{mn} \cos\left(c\sqrt{\lambda_{mn}}t\right) + b_{mn} \sin\left(c\sqrt{\lambda_{mn}}t\right) \right) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$



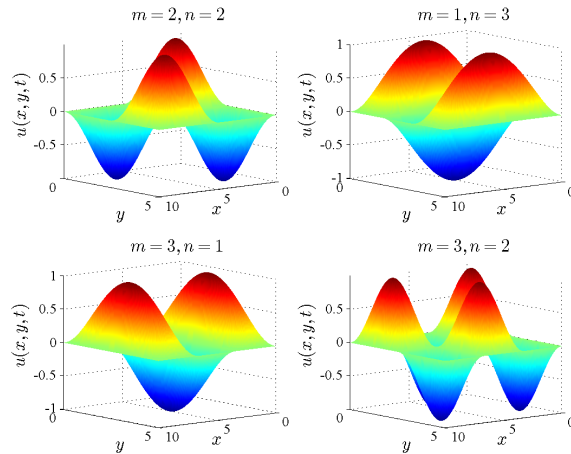
## Nodal Curves

### Nodal Curves



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## Rectangular Membrane

From the ICs, we have

$$u(x, y, 0) = \alpha(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

Multiply by  $\sin\left(\frac{j\pi x}{L}\right)$  and integrate  $x \in [0, L]$  and  $\sin\left(\frac{n\pi y}{H}\right)$  and integrate  $y \in [0, H]$ . **Orthogonality** gives:

$$a_{mn} = \frac{4}{LH} \int_0^H \int_0^L \alpha(x, y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) dx dy.$$

Similarly,

$$u_t(x, y, 0) = \beta(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} c\sqrt{\lambda_{mn}} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right),$$

and **orthogonality** gives:

$$b_{mn} = \frac{4}{LHc\sqrt{\lambda_{mn}}} \int_0^H \int_0^L \beta(x, y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right) dx dy.$$

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## Theorems for Eigenvalue Problems

### Helmholtz Equation:

$$\nabla^2 \phi + \lambda \phi = 0 \quad \text{in } R,$$

with

$$\alpha \phi + \beta \nabla \phi \cdot \mathbf{\hat{n}} = 0 \quad \text{on } \partial R.$$

Generalizes to

$$\nabla \cdot (p \nabla \phi) + q \phi + \lambda \sigma \phi = 0.$$

### Theorem

1. All **eigenvalues** are real.
2. There exists infinitely many **eigenvalues** with a smallest, but no largest **eigenvalue**.
3. There may be many **eigenfunctions** corresponding to an **eigenvalue**.

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## Theorems for Eigenvalue Problems

### Theorem

4. The **eigenfunctions** form a complete set, so if  $f(x, y)$  is **piecewise smooth**

$$f(x, y) \sim \sum_{\lambda} a_{\lambda} \phi_{\lambda}(x, y).$$

5. **Eigenfunctions** corresponding to different **eigenvalues** are orthogonal

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} \sigma dR = 0 \quad \text{if } \lambda_1 \neq \lambda_2.$$

Different **eigenfunctions** belonging to the same **eigenvalue** can be made **orthogonal** by **Gram-Schmidt process**.

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## Theorems for Eigenvalue Problems

### Theorem

6. For  $\sigma = 1$ , an **eigenvalue**  $\lambda$  can be related to the **eigenfunction** by the Rayleigh quotient:

$$\lambda = \frac{-\oint_{\partial R} \phi \nabla \phi \cdot n \, ds + \iint_R |\nabla \phi|^2 \, dR}{\iint_R \phi^2 \, dR}.$$

The boundary conditions often simplify the boundary integral.

We use the **Example** for the **vibrating rectangular membrane** to illustrate a number of the Theorem results above.



## Example

**Example:** The Sturm-Liouville problem for the **vibrating rectangular membrane** satisfies:

**PDE:**  $\nabla^2 \phi + \lambda \phi = 0,$

**ICs:**  $\phi(0, y) = 0, \quad \phi(L, y) = 0,$

$\phi(x, 0) = 0, \quad \phi(x, H) = 0.$

We have already shown that this **Helmholtz equation** has **eigenvalues**:

$$\lambda_{mn} = \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2, \quad m = 1, 2, \dots \quad n = 1, 2, \dots$$

with corresponding **eigenfunctions**:

$$\phi_{mn}(x, y) = \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right), \quad m = 1, 2, \dots \quad n = 1, 2, \dots$$



## Example

**Example (cont):** We already demonstrated that:

- 1 **Real eigenvalues:** The **eigenvalues** are clearly real.
- 2 **Ordering the eigenvalues:** It is easy to see that there is the lowest **eigenvalue**  $\lambda_1 = \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2$  and that there is no largest **eigenvalue**, as  $m$  or  $n \rightarrow \infty$ .
- 3 **Multiple eigenvalues:** Suppose that  $L = 2H$ . It follows that

$$\lambda_{mn} = \frac{\pi^2}{4H^2} (m^2 + 4n^2).$$

It is easy to see for  $m = 4, n = 1$  and  $m = 2, n = 2$ ,

$$\lambda_{41} = \lambda_{22} = \frac{5\pi^2}{H^2}.$$

These solutions will oscillate with the same frequency.



## Example

**Example (cont):** We have:

- 4 **Series of eigenfunctions:** If  $f(x, y)$  is **piecewise smooth**, then

$$f(x, y) \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{H}\right).$$

- 5 **Convergence:** As before, write the **Error** using a finite series

$$E = \iint_R \left( f - \sum_{\lambda} a_{\lambda} \phi_{\lambda} \right)^2 \, dR.$$

The approximation improves with increasing  $\lambda$ , and we found that the series  $\sum_{\lambda} a_{\lambda} \phi_{\lambda}$  **converges in the mean** to  $f$ .



## Orthogonality

**Orthogonality:** Assume  $\lambda_1 \neq \lambda_2$  with *eigenfunctions*  $\phi_{\lambda_1}$  and  $\phi_{\lambda_2}$  and insert these into the equation:

$$\nabla \cdot (p\nabla\phi) + q\phi + \lambda\sigma\phi = 0.$$

Multiplying by the other eigenfunction and subtracting, we can write

$$\phi_{\lambda_1} (\nabla \cdot (p\nabla\phi_{\lambda_2})) - \phi_{\lambda_2} (\nabla \cdot (p\nabla\phi_{\lambda_1})) = (\lambda_2 - \lambda_1)\sigma\phi_{\lambda_1}\phi_{\lambda_2}.$$

Use integration by parts over the entire region  $R$  and the homogeneous boundary conditions to give (more details next section):

$$\iint_R \phi_{\lambda_1}\phi_{\lambda_2}\sigma dR = 0, \quad \text{if } \lambda_1 \neq \lambda_2.$$



## Green's Formula

Consider the **PDE**:

$$\nabla^2\phi + \lambda\phi = 0, \quad \text{in } R,$$

with **BCs**:

$$\beta_1\phi + \beta_2\nabla\phi \cdot \mathbf{\hat{n}} = 0, \quad \text{on } \partial R,$$

where  $\beta_1$  and  $\beta_2$  are real functions in  $R$ .

Basic product rule gives:

$$\begin{aligned} \nabla \cdot (u\nabla v) &= u\nabla^2 v + \nabla u \cdot \nabla v, \\ \nabla \cdot (v\nabla u) &= v\nabla^2 u + \nabla v \cdot \nabla u. \end{aligned}$$

Subtracting gives:

$$u\nabla^2 v - v\nabla^2 u = \nabla \cdot (u\nabla v - v\nabla u).$$



## Fourier Coefficients

**Fourier Coefficients:** Assume that  $f$  is *piecewise smooth*, so

$$f(x, y) \sim \sum_{\lambda} a_{\lambda}\phi_{\lambda}.$$

Use the *orthogonality relationship* with respect to the weighting function  $\sigma$ :

$$\iint_R \phi_{\lambda_1}\phi_{\lambda_2}\sigma dR = 0, \quad \text{if } \lambda_1 \neq \lambda_2,$$

then the **Fourier coefficients** satisfy

$$a_{\lambda_i} = \frac{\iint_R f\phi_{\lambda_i}\sigma dR}{\iint_R \phi_{\lambda_i}^2\sigma dR}.$$

**Note:** If there is more than one *eigenfunction* associated with an *eigenvalue*, then assume the *eigenfunctions* have been made *orthogonal* by *Gram-Schmidt*.



## Green's Formula

The previous result is integrated to give:

$$\iint_R (u\nabla^2 v - v\nabla^2 u) dR = \iint_R \nabla \cdot (u\nabla v - v\nabla u) dR.$$

Apply the **Divergence Theorem** and obtain:

**Green's Formula:** Also, **Green's second identity:**

$$\iint_R (u\nabla^2 v - v\nabla^2 u) dR = \oint_{\partial R} (u\nabla v - v\nabla u) \cdot \mathbf{\hat{n}} dS.$$

This identity is important in showing an operator is **self-adjoint** if there are *homogeneous BCs*.



## Self-Adjoint Operator

Let  $L = \nabla^2$  be a linear operator:

### Theorem (Self-Adjoint)

If  $u$  and  $v$  are two functions such that

$$\oint_{\partial R} (u \nabla v - v \nabla u) \cdot \tilde{\mathbf{n}} \, dS = 0,$$

then

$$\iint_R (u \nabla^2 v - v \nabla^2 u) \, dR = \iint_R (u L[v] - v L[u]) \, dR = 0.$$

**Note:** The above theorem is stated in 2D, but it equally applies to 3D by substituting double integrals with triple integrals and line integrals with surface integrals.



## Orthogonality

However,

$$\begin{aligned} \iint_R (\phi_1 L[\phi_2] - \phi_2 L[\phi_1]) \, dR &= \iint_R (\lambda_2 \phi_1 \phi_2 - \lambda_1 \phi_1 \phi_2) \, dR \\ &= (\lambda_2 - \lambda_1) \iint_R \phi_1 \phi_2 \, dR = 0. \end{aligned}$$

So for  $\lambda_2 \neq \lambda_1$ , the *eigenfunctions* are **orthogonal**:

$$\iint_R \phi_1 \phi_2 \, dR = 0.$$



## Orthogonality

**Orthogonality of Eigenfunctions:** We use **Green's formula** to show *orthogonality* of *eigenfunctions*,  $\phi_1$  and  $\phi_2$ , corresponding to different *eigenvalues*,  $\lambda_1$  and  $\lambda_2$ .

Suppose with  $L = \nabla^2$

$$L[\phi_1] + \lambda_1 \phi_1 = 0 \quad \text{and} \quad L[\phi_2] + \lambda_2 \phi_2 = 0.$$

If  $\phi_1$  and  $\phi_2$  satisfy the same *homogeneous BCs*,

$$\oint_{\partial R} (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot \tilde{\mathbf{n}} \, dS = 0,$$

then by **Green's formula**:

$$\iint_R (\phi_1 L[\phi_2] - \phi_2 L[\phi_1]) \, dR = 0.$$



## Gram-Schmidt Process

**Gram-Schmidt Process:** Suppose that  $\phi_1, \phi_2, \dots, \phi_m$ , are independent *eigenfunctions* all corresponding to the *eigenvalue*,  $\lambda$  (a **single e.v.**).

Let  $\psi_1 = \phi_1$  be an *eigenfunction*.

Any linear combination of *eigenfunctions* is also an *eigenfunction*, so take

$$\psi_2 = \phi_2 + c\psi_1.$$

We want

$$\iint_R \psi_1 \psi_2 \, dR = 0 = \iint_R \psi_1 (\phi_2 + c\psi_1) \, dR,$$

so choose

$$c = -\frac{\iint_R \phi_2 \psi_1 \, dR}{\iint_R \psi_1^2 \, dR}.$$



## Gram-Schmidt Process

**Gram-Schmidt Process:** Continuing take

$$\psi_3 = \phi_3 + c_1\psi_1 + c_2\psi_2.$$

We want

$$\begin{aligned} \iint_R \psi_3 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} dR &= 0, \\ \iint_R (\phi_3 + c_1\psi_1 + c_2\psi_2) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} dR &= 0. \end{aligned}$$

It follows that

$$c_1 = -\frac{\iint_R \phi_3 \psi_1 dR}{\iint_R \psi_1^2 dR} \quad \text{and} \quad c_2 = -\frac{\iint_R \phi_3 \psi_2 dR}{\iint_R \psi_2^2 dR}.$$



## Gram-Schmidt Process

**Gram-Schmidt Process:** In general,

$$\psi_j = \phi_j - \sum_{i=1}^{j-1} \frac{\iint_R \phi_j \psi_i dR}{\iint_R \psi_i^2 dR} \psi_i.$$

Thus, we can always obtain an *orthogonal set of eigenfunctions*.

