Math 531 - Partial Differential Equations

Heat Conduction — in Higher Dimensions

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Outline

Heat Conduction in a Higher Dimensions

Previously we developed the ***heat equation*** for a one-dimensional rod

We want to extend the ***heat equation*** for higher dimensions

Conservation of Heat Energy: In any volume element, the basic conservation equation for **heat** satisfies

Rate of change of heat energy in time	=	Heat energy flowing across boundaries per unit time	+	Heat energy generated inside per unit time

Define $c(x, y, z)$ to be the ***specific heat of a material*** (the ***heat energy*** required to raise a unit mass of a material a unit of temperature)

Define $\rho(x, y, z)$ to be the ***mass density*** (per unit volume)

Define $u(x, y, z, t)$ as the ***temperature of a material***

Heat Conduction in a Higher Dimensions

The specific heat, c, mass density, ρ, and temperature, u, are used with the conservation law above to create the general ***heat equation***

The total energy in a volume element R satisfies:

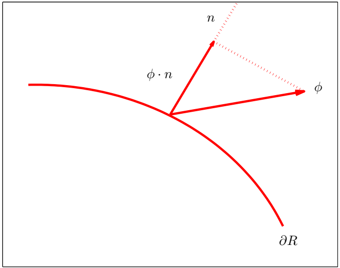
$$\text{Total Energy} = \iiint_R c\rho u \, dV.$$

The rate of change of heat energy in time is given by

$$\frac{d}{dt} \iiint_R c\rho u \, dV.$$

Heat Conduction in a Higher Dimensions

Define $\phi(x, y, z, t)$ as the ***heat flux vector*** for the heat crossing the surface of the region R denoted ∂R, and define n as the ***outward normal vector***



By convention the heat flux is the flow directed into the region R, so the ***heat flux*** into the region R is the integral over ∂R of $-\phi \cdot n$.

$$-\oiint_{\partial R} \phi \cdot n \, dS$$

Heat Conduction in a Higher Dimensions

Define $Q(x, y, z, t)$ as the ***heat energy*** generated per unit time from the ***sources*** or ***sinks*** inside the region R.

This gives

$$\iiint_R Q\,dV.$$

The **Conservation of Heat Energy** combines these terms to give:

$$\frac{d}{dt}\iiint_R c\rho u\,dV = -\oiint_{\partial R} \phi \cdot n\,dS + \iiint_R Q\,dV.$$

We need to combine these terms to obtain the general **Heat Equation**.

Heat Conduction in a Higher Dimensions

Theorem (Divergence or Gauss's Theorem)

Suppose R is a subset of $\mathbb{R}^3$, which is compact and has a piecewise smooth boundary ∂R. If ϕ is a continuously differentiable vector field defined on a neighborhood of R, then we have:

$$\oiint_{\partial R} (\phi \cdot n)\, dS = \iiint_R (\nabla \cdot \phi)\, dV.$$

The **Conservation of Heat Energy** combines these terms to give:

$$\frac{d}{dt} \iiint_R c\rho u\, dV = -\iiint_R (\nabla \cdot \phi)\, dV + \iiint_R Q\, dV.$$

Heat Conduction in a Higher Dimensions

The previous equation is rearranged to give:

$$\iiint_R \left(c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q \right) dV = 0.$$

Since this holds for any region R, we have the **heat equation**:

$$c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \phi + Q.$$

Fourier's law of heat conduction satisfies:

$$\phi = -K_0 \nabla u,$$

which produces the **heat equation** in higher dimensions:

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q.$$

Heat Equation in a Higher Dimensions

The **heat equation** in higher dimensions is:

$$c\rho\frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q.$$

If the Fourier coefficient is constant, K_0, as well as the specific heat, c, and material density, ρ, and if there are no sources or sinks, $Q \equiv 0$, then the **heat equation** becomes

$$\frac{\partial u}{\partial t} = k\nabla^2 u, \qquad t > 0 \quad \text{and} \quad (x, y, z) \in R,$$

where $k = K_0/(c\rho)$ and

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

in Cartesian coordinates.

Poisson's and Laplace's Equations

The **heat equation** in higher dimensions is:

$$c\rho\frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q.$$

If the Fourier coefficient is constant, K_0, then the **Steady-State** problem can be written:

$$\nabla^2 u = -\frac{Q}{K_0},$$

which is **Poisson's equation**

Furthermore, if there are no sources or sinks $(Q \equiv 0)$, then we obtain **Laplace's equation**

$$\nabla^2 u = 0.$$

Laplacian in 2D

In Cartesian coordinates,the Laplacian in 2D is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Recall that in polar coordinates

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta).$$

By using the **chain rule** and the **dot product**, we find:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Laplacian in 3D

In cylindrical coordinates

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad \text{and} \quad z = z,$$

so

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

In spherical coordinates

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad \text{and} \quad z = \rho\cos(\phi),$$

so it can be shown (HW exercise):

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2\sin^2(\phi)}\frac{\partial^2 u}{\partial \theta^2}.$$