Math 531 - Partial Differential Equations
Fourier Transforms for PDEs - Part C

Joseph M. Mahaffy,
⟨jmahaffy@mail.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

Spring 2023

Outline

Heat Equation on Semi-Infinite Domains

Consider the **PDE** for the ***heat equation*** on a semi-infinite domain:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t>0, \quad x>0,$$

with the **BC** and **IC**:

$$u(0,t)=0 \qquad \text{and} \qquad u(x,0)=f(x),$$

where we assume $f(x)\to 0$ as $x\to\infty$.

We employ the ***separation of variables***, $u(x,t)=h(t)\phi(x)$, where the ***Sturm-Liouville problem*** is

$$\phi''+\lambda\phi=0, \qquad \phi(0)=0 \quad \text{and} \quad \lim_{x\to\infty}|\phi(x)|<\infty.$$

The solution to the SL-Problem is:

$$\phi(x)=c_1\sin(\omega x), \qquad \text{where} \quad \omega=\sqrt{\lambda}.$$

Heat Equation on Semi-Infinite Domains

The ODE in t is $h'=-k\omega^2 h$, which has the solution

$$h(t)=c\,e^{-k\omega^2 t}.$$

Thus, the ***product solution*** becomes

$$u_\omega(x,t)=A(\omega)\sin(\omega x)e^{-k\omega^2 t}, \qquad \omega>0.$$

The ***superposition principle*** gives the solution:

$$u(x,t)=\int_0^\infty A(\omega)\sin(\omega x)e^{-k\omega^2 t}\,d\omega,$$

where

$$f(x)=\int_0^\infty A(\omega)\sin(\omega x)\,d\omega,$$

and

$$A(\omega)=\frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)dx.$$

Fourier Sine Transform

From the ***Fourier transforms*** with complex exponentials, we have the ***Fourier pair***:

$$f(x) = \frac{1}{\gamma}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega,$$
$$F(\omega) = \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx, \qquad \text{for any } \gamma.$$

If $f(x)$ is odd (choose an odd extension),

$$F(\omega) = \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} f(x)\left(\cos(\omega x) + i\sin(\omega x)\right)\,dx,$$
$$= \frac{2i\gamma}{2\pi}\int_{0}^{\infty} f(x)\sin(\omega x)\,dx.$$

Note $F(\omega)$ is an odd function of ω, so

$$f(x) = \frac{1}{\gamma}\int_{-\infty}^{\infty} F(\omega)\left(\cos(\omega x) - i\sin(\omega x)\right)\,d\omega,$$
$$= -\frac{2i}{\gamma}\int_{0}^{\infty} F(\omega)\sin(\omega x)\,d\omega,$$

Fourier Sine and Cosine Transforms

For convenience, take $-\frac{2i}{\gamma} = 1$, so for $f(x)$ odd we obtain the ***Fourier sine transform pair***:

$$f(x) = \int_0^{\infty} F(\omega)\sin(\omega x)\,d\omega \equiv S^{-1}[F(\omega)],$$
$$F(\omega) = \frac{2}{\pi}\int_0^{\infty} f(x)\sin(\omega x)\,dx \equiv S[f(x)].$$

Note that some like to have symmetry and have a coefficient in front of the integrals as $\sqrt{2/\pi}$.

If $f(x)$ is even, then we obtain the ***Fourier cosine transform pair***:

$$f(x) = \int_0^{\infty} F(\omega)\cos(\omega x)\,d\omega \equiv C^{-1}[F(\omega)],$$
$$F(\omega) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos(\omega x)\,dx \equiv C[f(x)].$$

Differentiation Rules for Sine and Cosine Transforms

Assume that both $f(x)$ and $\frac{df}{dx}(x)$ are continuous and both are vanishing for large x, *i.e.*, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty}\frac{df}{dx}(x) = 0$.

Use integration by parts to find the transforms of the first derivatives:

$$C\left[\frac{df}{dx}\right] = \frac{2}{\pi}\int_0^{\infty}\frac{df}{dx}\cos(\omega x)\,dx = \frac{2}{\pi}f(x)\cos(\omega x)\Big|_0^{\infty} + \frac{2\omega}{\pi}\int_0^{\infty} f(x)\sin(\omega x)\,dx,$$

and

$$S\left[\frac{df}{dx}\right] = \frac{2}{\pi}\int_0^{\infty}\frac{df}{dx}\sin(\omega x)\,dx = \frac{2}{\pi}f(x)\sin(\omega x)\Big|_0^{\infty} - \frac{2\omega}{\pi}\int_0^{\infty} f(x)\cos(\omega x)\,dx.$$

It follows that

$$C\left[\frac{df}{dx}\right] = -\frac{2}{\pi}f(0) + \omega S[f]$$

and

$$S\left[\frac{df}{dx}\right] = -\omega C[f].$$

Note that these formulas imply that if the PDE has any first partial w.r.t. the potential transformed variable, then ***Fourier sine*** or ***Fourier cosine transforms*** won't work.

Differentiation Rules for Sine and Cosine Transforms

From the pair,

$$C\left[\frac{df}{dx}\right] = -\frac{2}{\pi}f(0) + \omega S[f]$$

and

$$S\left[\frac{df}{dx}\right] = -\omega C[f],$$

we can readily obtain the transforms of the second derivatives:

$$C\left[\frac{d^2f}{dx^2}\right] = -\frac{2}{\pi}\frac{df}{dx}(0) + \omega S\left[\frac{df}{dx}\right] = -\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2 C[f]$$

and

$$S\left[\frac{d^2f}{dx^2}\right] = -\omega C\left[\frac{df}{dx}\right] = \frac{2}{\pi}\omega f(0) - \omega^2 S[f].$$

Note: When solving a PDE (with second partials), then either $f(0)$ must be known and ***Fourier sine transforms*** are used or $\frac{df}{dx}(0)$ must be known and ***Fourier cosine transforms*** are used.

Heat Equation on Semi-Infinite Domain

Consider the **PDE** for the ***heat equation*** on a semi-infinite domain:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad x > 0,$$

with the **BC** and **IC**:

$$u(0,t) = g(t) \qquad \text{and} \qquad u(x,0) = f(x).$$

Since the **BC** is nonhomogeneous, the technique of ***separation of variables*** does NOT apply.

Since we know u at $x = 0$, we want to apply the ***Fourier sine transform*** to the PDE.

Heat Equation on Semi-Infinite Domain

For the nonhomogeneous equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad x > 0,$$

we apply the ***Fourier sine transform***:

$$\overline{U}(\omega,t) = \frac{2}{\pi}\int_0^\infty u(x,t)\sin(\omega x)\,dx,$$

which gives the **ODE** in $\overline{U}$

$$\frac{\partial \overline{U}}{\partial t} = k\left(\frac{2}{\pi}\omega g(t) - \omega^2\overline{U}\right).$$

The ***Fourier sine transform*** of the initial condition is:

$$\overline{U}(\omega,0) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx.$$

Heat Equation on Semi-Infinite Domain

The **ODE** is linear and can be written:

$$\frac{\partial \overline{U}}{\partial t} + k\omega^2\overline{U} = \frac{2k\omega}{\pi}g(t),$$

which is readily solved to give:

$$\overline{U}(\omega,t) = \overline{U}(\omega,0)e^{-k\omega^2 t} + \frac{2k\omega}{\pi}\int_0^t e^{-k\omega^2(t-s)}g(s)\,ds.$$

This problem is readily solved with programs similar to the ones shown earlier.

With specific **ICs**, $f(x)$, and **BCs**, $g(t)$, the integrals can be formed, then numerically computed.

Heat Equation on Semi-Infinite Domain

As a specific example, we choose to numerically show the solution with

$$u(x,0) = f(x) = 0, \qquad \text{and} \qquad u(0,t) = g(t) = e^{-at}.$$

The ***Fourier sine transform*** satisfies:

$$\begin{aligned}\overline{U}(\omega,t) &= \overline{U}(\omega,0)e^{-k\omega^2 t} + \frac{2k\omega}{\pi}\int_0^t e^{-k\omega^2(t-s)}g(s)\,ds,\\ \overline{U}(\omega,t) &= \frac{2k\omega}{\pi}\frac{\left(e^{-k\omega^2 t} - e^{-at}\right)}{a - k\omega^2}.\end{aligned}$$

It follows that

$$u(x,t) = \int_0^\infty \overline{U}(\omega,t)\sin(\omega x)\,d\omega.$$

Heat Equation on Semi-Infinite Domain

Enter the **Maple** commands for the graph of $u(x,t)$

```
u := (x,t) -> (2/Pi)*(int(w*(exp(-w^2*t)-exp(-0.1*t))*sin(w*x)/
              (0.1-w^2), w = 0..50));
plot3d(u(x,t), x = 0..10, t = 0..20);
```

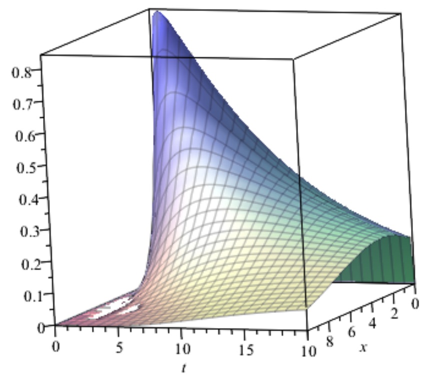
The **IC** is

$$f(x) = 0.$$

The **BC** is

$$g(t) = e^{-0.1t}.$$

This graph shows the ***diffusion*** of the heat with time.



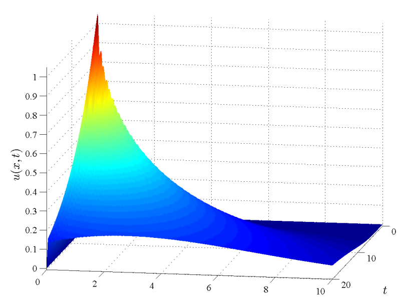
Heat Equation on Semi-Infinite Domain

In both **Maple** and **MatLab**, the integral over ω is truncated at 50. The figure below shows that this creates some oscillations.

```
% Solution Heat Equation with FT
% f(x) = 0, u(0,t) = e^(-t)
N1 = 201; N2 = 201;
tv = linspace(0,20,N1);
xv = linspace(0,10,N2);
[t1,x1] = ndgrid(tv,xv);
f = @(w,c) (2*w/pi).*(exp(-c(1)*w.^2)-...
    exp(-0.1*c(1)))./(0.1-w.^2);
for i = 1:N1
    for j = 1:N2
        c = [t1(i,j),x1(i,j)];
        U(i,j) = ...
            integral(@(w)f(w,c).*sin(w*c(2)),0,50);
    end
end
```


Heat Equation on Semi-Infinite Domain

```
set(gca,'FontSize',[12]);
surf(t1,x1,U);
shading interp
colormap(jet)
view([100 15])
```



Wave Equation

Consider the ***wave equation*** on an infinite domain:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$

with the **ICs**:

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0,$$

where the latter **IC** is to simplify the problem.

The ***Fourier transform pair*** satisfies:

$$\begin{aligned} \overline{U}(\omega,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\,dx, \\ u(x,t) &= \int_{-\infty}^{\infty} \overline{U}(\omega,t)e^{i\omega x}\,d\omega. \end{aligned}$$

Wave Equation

From the ***differentiation rules***, we have

$$\frac{\partial^2 \overline{U}}{\partial t^2} = -c^2\omega^2\overline{U},$$

where the **ICs** give

$$\begin{aligned}\overline{U}(\omega,0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx,\\ \frac{\partial \overline{U}(\omega,0)}{\partial t} &= 0.\end{aligned}$$

The general solution becomes:

$$\overline{U}(\omega,t) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t).$$

The **IC** with the velocity being **zero** gives $B(\omega) = 0$.

Wave Equation

The ***initial position*** gives:

$$A(\omega) = \overline{U}(\omega,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx.$$

The ***inverse Fourier transform*** satisfies:

$$u(x,t) = \int_{-\infty}^{\infty} \overline{U}(\omega,0)\cos(c\omega t)e^{-i\omega x}\,d\omega.$$

Euler's formula gives $\cos(c\omega t) = \frac{e^{ic\omega t}+e^{-ic\omega t}}{2}$, so

$$u(x,t) = \int_{-\infty}^{\infty} \overline{U}(\omega,0)\left[\frac{e^{-i\omega(x-ct)}+e^{-i\omega(x+ct)}}{2}\right]d\omega.$$

Wave Equation

Since

$$f(x) = \int_{-\infty}^{\infty} \overline{U}(\omega,0)e^{-i\omega x}\,d\omega,$$

we have

$$\begin{aligned}u(x,t) &= \int_{-\infty}^{\infty} \overline{U}(\omega,0)\left[\frac{e^{-i\omega(x-ct)}+e^{-i\omega(x+ct)}}{2}\right]d\omega,\\ u(x,t) &= \frac{1}{2}\left[f(x-ct)+f(x+ct)\right].\end{aligned}$$

It follows that the ***initial position*** breaks into **2** ***traveling waves*** with velocity c in opposite directions.

This solution is also obtained using ***D'Alembert's method***.

Laplace's Equation on a Semi-Infinite Strip

Consider ***Laplace's equation on a semi-infinite strip***:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < L, \quad y > 0.$$

with **BCs**:

$$u(0,y) = g_1(y), \qquad u(L,y) = g_2(y), \qquad u(x,0) = f(x).$$

Divide the problem into

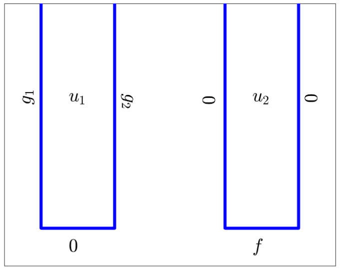
$$\nabla^2 u_1 = 0,$$

with ***homogeneous BC*** on the bottom.

Second problem is

$$\nabla^2 u_2 = 0,$$

with ***homogeneous BCs*** on the sides.



Laplace's Equation on a Semi-Infinite Strip

Consider Laplace's equation

$$\nabla^2 u_2 = 0, \qquad 0 < x < L, \quad y > 0,$$

with **BCs**:

$$u_2(0,y) = 0, \quad u_2(L,y) = 0, \quad \text{and} \quad u_2(x,0) = f(x).$$

Separation of variables with $u(x,y) = \phi(x)h(y)$ gives

$$\frac{\phi''}{\phi} = -\frac{h''}{h} = -\lambda, \qquad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

The ***Sturm-Liouville problem*** is

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0,$$

so the ***eigenvalues*** and ***eigenfunctions*** are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \text{and} \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Laplace's Equation on a Semi-Infinite Strip

The other **ODE** is $h'' - \lambda_n h = 0$, which has the solution:

$$h_n(y) = c_1 e^{-\frac{n\pi y}{L}} + c_2 e^{\frac{n\pi y}{L}}.$$

For the $h_n(y)$ to be bounded as $y \to \infty$, then $c_2 = 0$.

The ***superposition principle*** gives

$$u_2(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n\pi y}{L}}.$$

The lower **BC**, $u(x,0) = f(x)$ gives

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Laplace's Equation on a Semi-Infinite Strip

The second Laplace's problem is:

$$\nabla^2 u_1 = 0, \qquad 0 < x < L, \quad y > 0,$$

with **BCs**:

$$u_1(0,y) = g_1(y), \quad u_1(L,y) = g_2(y), \quad \text{and} \quad u_1(x,0) = 0.$$

Separation of variables for this case gives

$$h(y) = c_1\cos(\omega y) + c_2\sin(\omega y), \qquad \text{for} \quad \omega \geq 0.$$

The ***homogeneous BC*** at $y = 0$ gives $c_1 = 0$, suggesting that we use the ***Fourier sine transform***.

Laplace's Equation on a Semi-Infinite Strip

The ***Fourier sine transform pair*** is:

$$\begin{aligned} u_1(x,y) &= \int_0^{\infty} \overline{U}_1(x,\omega)\sin(\omega y)\,d\omega, \\ \overline{U}_1(x,\omega) &= \frac{2}{\pi}\int_0^{\infty} u_1(x,y)\sin(\omega y)\,dy. \end{aligned}$$

Recall

$$S\left[\frac{\partial^2 u_1}{\partial y^2}\right] = \frac{2}{\pi}\omega u_1(x,0) - \omega^2 S[u_1].$$

Laplace's equation becomes:

$$\frac{\partial^2 \overline{U}_1}{\partial x^2} - \omega^2\overline{U}_1 = 0,$$

which is easily solved.

Laplace's Equation on a Semi-Infinite Strip

It is convenient to take the solution of the form:

$$\overline{U}_1(x,\omega) = a(\omega)\sinh(\omega x) + b(\omega)\sinh(\omega(L-x)).$$

The **BCs** give:

$$\begin{aligned}\overline{U}_1(0,\omega) &= b(\omega)\sinh(\omega L) = \frac{2}{\pi}\int_0^\infty g_1(y)\sin(\omega y)\,dy,\\ \overline{U}_1(L,\omega) &= a(\omega)\sinh(\omega L) = \frac{2}{\pi}\int_0^\infty g_2(y)\sin(\omega y)\,dy,\end{aligned}$$

so we can readily find $a(\omega)$ and $b(\omega)$,

$$a(\omega) = \frac{2}{\pi\sinh(\omega L)}\int_0^\infty g_2(y)\sin(\omega y)\,dy \quad \text{and} \quad b(\omega) = \frac{2}{\pi\sinh(\omega L)}\int_0^\infty g_1(y)\sin(\omega y)\,dy.$$

Laplace's Equation on a Semi-Infinite Strip

Example: Consider the specific case:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < 2, \quad y > 0.$$

with **BCs**:

$$u(0,y) = e^{-y}\sin(y), \qquad u(2,y) = \begin{cases} 2, & y < 5, \\ 0, & y > 5, \end{cases} \qquad u(x,0) = x.$$

This problem is broken into the **2** problems with either a homogeneous end condition or homogeneous side conditions, then the **2** solutions are added together.

We provide the details to produce a temperature profile for this problem, using the previous work.

Laplace's Equation on a Semi-Infinite Strip

When the two sides are homogeneous,

$$\nabla^2 u_2 = 0, \qquad 0 < x < 2, \quad y > 0,$$

with **BCs**:

$$u_2(0,y) = 0, \quad u_2(2,y) = 0, \quad \text{and} \quad u_2(x,0) = x.$$

From before, the solution is:

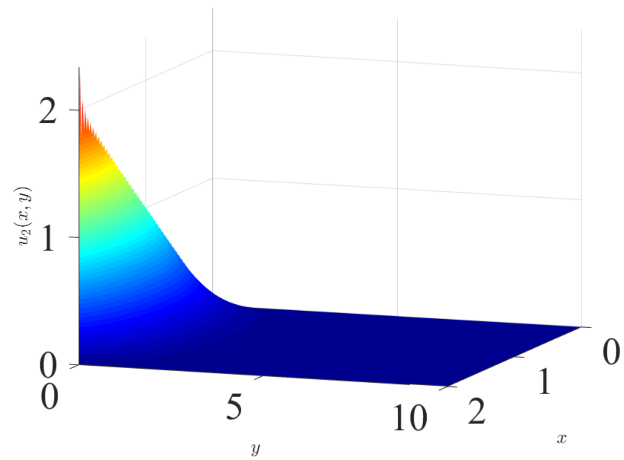
$$u_2(x,y) = \sum_{n=1}^\infty a_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{n\pi y}{2}},$$

where using **Maple**, we find:

$$a_n = \int_0^2 x\sin\left(\frac{n\pi x}{2}\right)\,dx = \frac{4(-1)^{n+1}}{n\pi}.$$

Laplace's Equation on a Semi-Infinite Strip

The ***steady-state temperature*** temperature profile for $u_2(x,y)$ using 100 terms in the series is shown below.



Laplace's Equation on a Semi-Infinite Strip

```
% Solution Laplace's equation - semi-infinite strip
N1 = 201; N2 = 201; M = 100;
xv = linspace(0,2,N1);
yv = linspace(0,10,N2);
[x1,y1] = ndgrid(xv,yv);
for i = 1:N1
    for j = 1:N2
        c = [x1(i,j),y1(i,j)];
        U2(i,j) = 0;
        for k = 1:M
            U2(i,j) = U2(i,j) + ...
                (4*(-1)^(k+1)/(k*pi))...
                *sin(k*pi*c(1)/2)*exp(-k*pi*c(2)/2);
        end
    end
end
```


Laplace's Equation on a Semi-Infinite Strip

Laplace's problem for $u_1(x,y)$ is:

$$\nabla^2 u_1 = 0, \qquad 0 < x < 2, \quad y > 0,$$

with **BCs**:

$$u_1(0,y) = e^{-y}\sin(y), \qquad u_1(2,y) = \begin{cases} 2, & y < 5, \\ 0, & y > 5, \end{cases} \qquad u_1(x,0) = 0.$$

From before, the ***Fourier transform solution*** satisfies:

$$u_1(x,y) = \int_0^\infty \overline{U}_1(x,\omega)\sin(\omega y)\,d\omega,$$

where

$$\overline{U}_1(x,\omega) = a(\omega)\sinh(\omega x) + b(\omega)\sinh(\omega(2-x)).$$

Laplace's Equation on a Semi-Infinite Strip

Once again **Maple** is used to find the coefficients $a(\omega)$ and $b(\omega)$:

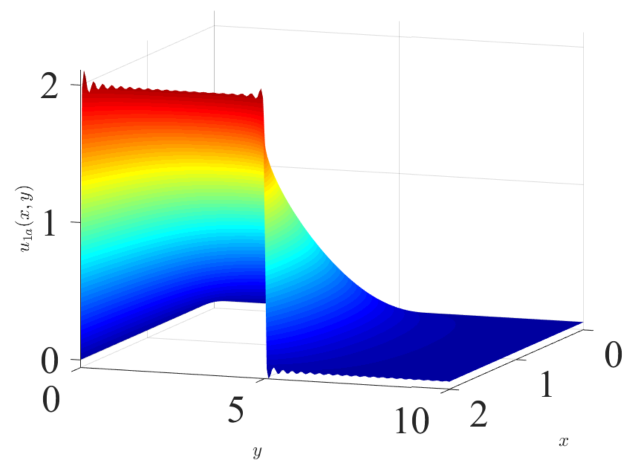
$$\begin{aligned} a(\omega) &= \frac{2}{\pi\sinh(2\omega)}\int_0^5 2\sin(\omega y)\,dy, \\ &= \frac{4(1-\cos(5\omega))}{\pi\omega\sinh(2\omega)}, \end{aligned}$$

and

$$\begin{aligned} b(\omega) &= \frac{2}{\pi\sinh(2\omega)}\int_0^\infty e^{-y}\sin(y)\sin(\omega y)\,dy, \\ &= \frac{4\omega}{\pi(\omega^2-2\omega+2)(\omega^2+2\omega+2)\sinh(2\omega)}. \end{aligned}$$

Laplace's Equation on a Semi-Infinite Strip

The ***steady-state temperature*** temperature profile for $u_{1a}(x,y)$ integrating on $\omega \in [0,100]$, where this only accounts for the **BC** at $x = 2$ ($b(\omega) = 0$), is shown below.



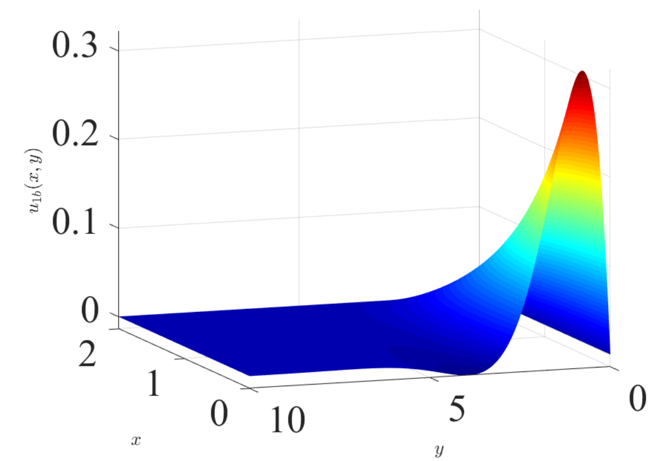
Laplace's Equation on a Semi-Infinite Strip

Below is the **MatLab** for the first part of $u_1(x,y)$

```
wmax = 100;
f = @(w,c) ...
     4*(1-cos(5*w)).*sinh(c(1)*w).*sin(c(2)*w)...
     ./(pi*w.*sinh(2*w));
for i = 1:N1
     for j = 1:N2
          c = [x1(i,j),y1(i,j)];
          U1a(i,j) = integral(@(w)f(w,c),0,wmax);
     end
end
surf(x1,y1,U1a);
shading interp
colormap(jet)
```


Laplace's Equation on a Semi-Infinite Strip

The ***steady-state temperature*** temperature profile for $u_{1b}(x,y)$ integrating on $\omega \in [0,100]$, where this only accounts for the **BC** at $x=0$ ($a(\omega)=0$), is shown below.



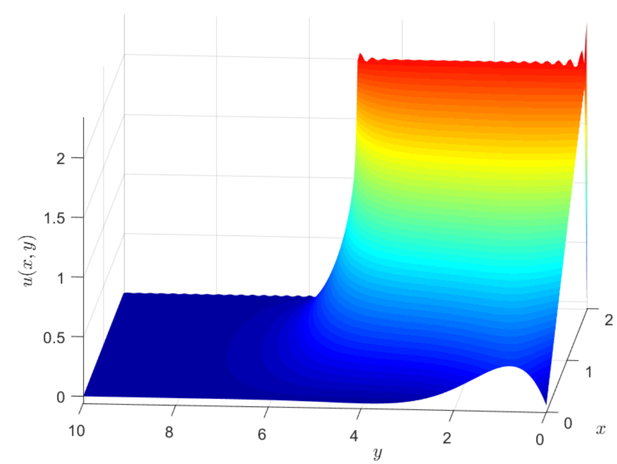
Laplace's Equation on a Semi-Infinite Strip

Below is the **MatLab** for the second part of $u_1(x,y)$

```
wmax = 100;
f = @(w,c) 4*w.*sinh((2-c(1))*w).*sin(c(2)*w)...
     ./(pi*(w.^2-2*w+2).*(w.^2+2*w+2).*sinh(2*w));
for i = 1:N1
     for j = 1:N2
          c = [x1(i,j),y1(i,j)];
          U1b(i,j) = integral(@(w)f(w,c),0,wmax);
     end
end
surf(x1,y1,U1b);
shading interp
colormap(jet)
```


Laplace's Equation on a Semi-Infinite Strip

Combining all the results above, the ***steady-state temperature*** temperature profile for $u(x,y)$ with the limits on number of terms in the series and the wave numbers ω in the integral is shown below.



Laplace's Equation on a Semi-Infinite Strip

Below is the **MatLab** for the complete ***steady-state temperature*** profile $u(x, y)$

```
for i = 1:N1
    for j = 1:N2
        U(i,j) = U2(i,j)+U1a(i,j)+U1b(i,j);
    end
end
surf(x1,y1,U);
shading interp
colormap(jet)
```