Math 531 - Partial Differential Equations

Fourier Transforms for PDEs - Part B

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Outline

1. Heat Equation and Fourier Transforms
   - Fundamental Solution and $\delta(x)$
   - Example

2. Fourier Transforms of Derivatives
   - Heat Equation
   - Convolution

Heat Equation and Fourier Transforms

We showed that $e^{-i\omega x}e^{-k\omega^2 t}$ solve the ***heat equation***, $u_t = ku_{xx}$, so

$$u(x,t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t}\, d\omega.$$

The **IC** is satisfied if:

$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x}\, d\omega.$$

From the definition of the ***Fourier transform***, the above equation is a ***Fourier integral*** representation of $f(x)$ with

$$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx.$$

Heat Equation and Fourier Transforms

The ***Fourier coefficient*** can be inserted into the solution:

$$u(x,t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{i\omega s}\, ds \right] e^{-i\omega x} e^{-k\omega^2 t}\, d\omega.$$

Interchanging the order of integration gives:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \left[\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega(x-s)}\, d\omega \right] ds.$$

However, the ***inverse Fourier transform*** of $e^{-k\omega^2 t}$

$$g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x}\, d\omega = \sqrt{\frac{\pi}{kt}} e^{-x^2/4kt}.$$

Heat Equation and Fourier Transforms

We insert the information above into the solution and obtain:

$$u(x,t) = \int_{-\infty}^{\infty} f(s) \left[\frac{1}{\sqrt{4\pi kt}} e^{-(x-s)^2/4kt} \right] ds.$$

It follows that each initial temperature "***influences***" the temperature at time t according to the ***Influence function***, which is related to the ***Green's functions*** last section:

$$G(x,t;s,0) = \frac{1}{\sqrt{4\pi kt}} e^{-(x-s)^2/4kt}.$$

This ***Influence function*** has problems near $t = 0$.

Dirac Delta function, $\delta(x)$

Define the function:

$$f(x,a) = \begin{cases} 0, & |x| > a, \\ \frac{1}{2a}, & |x| < a. \end{cases}$$

The ***Dirac delta function*** satisfies:

$$\lim_{a \to 0} f(x,a) = \delta(x).$$

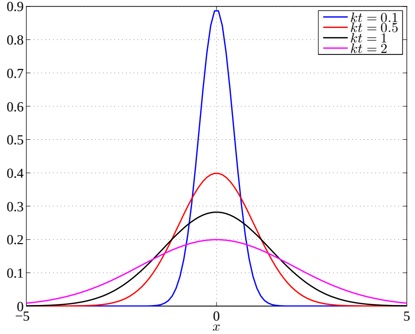
With regards to our ***Heat problem***, we see that as $t \to 0$ the ***influence*** is concentrated locally:

$$\lim_{t \to 0^+} \frac{1}{\sqrt{4\pi kt}} e^{-(x-s)^2/4kt} = \delta(x-s).$$

Fundamental Solution

Fundamental Solution: Suppose all the heat is concentrated at the origin, $u(x,0) = \delta(x)$, then

$$u(x,t) = \int_{-\infty}^{\infty} \delta(s) \left[\frac{1}{\sqrt{4\pi kt}} e^{-(x-s)^2/4kt} \right] ds = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}.$$



Heat Equation and Fourier Transforms

Example: Consider the infinite rod satisfying the ***heat equation***:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad -\infty < x < \infty,$$

with **IC**

$$u(x,0) = f(x) = \begin{cases} 0, & x < 0, \\ 100, & x > 0. \end{cases}$$

From above the solution satisfies:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} f(s)\left[\frac{1}{\sqrt{4\pi kt}}e^{-(x-s)^2/4kt}\right] ds, \\ &= \frac{100}{\sqrt{4\pi kt}}\int_0^{\infty} e^{-(x-s)^2/4kt}\, ds. \end{aligned}$$

Heat Equation and Fourier Transforms

With the change of dummy variables in the integral, $z = (s-x)/\sqrt{4kt}$, the solution can be written:

$$\begin{aligned} u(x,t) &= \frac{100}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x-s)^2/4kt}\,ds, \\ &= \frac{100}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^\infty e^{-z^2}\,dz, \\ &= \frac{100}{\sqrt{\pi}} \left[\int_0^\infty e^{-z^2}\,dz + \int_0^{x/\sqrt{4kt}} e^{-z^2}\,dz \right], \end{aligned}$$

by the evenness of e^{-z^2}.

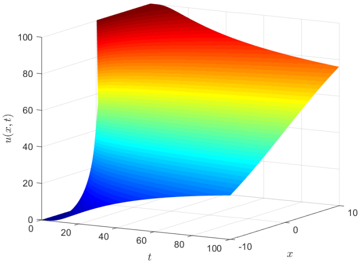
Thus, we can write the solution:

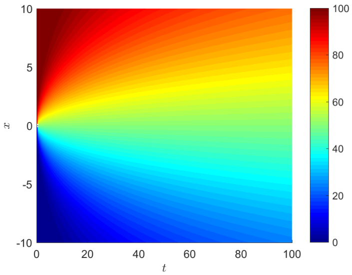
$$\begin{aligned} u(x,t) &= 50 + \frac{100}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-z^2}\,dz, \\ &= 50 \left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}} \right) \right). \end{aligned}$$

Heat Equation and Fourier Transforms

The ***temperature*** spreads by ***diffusion***.

The thermal energy spreads with infinite propagation speed.





Heat Equation and Fourier Transforms

Below is the MatLab code for the previous figures for **Heat Propagation**.

```
% Solutions to the heat flow equation
% on one-dimensional rod
% Fourier Transform solution
format compact;
tfin = 100;             % final time
xwid = 10;
k = 1;                  % heat capacitance
NptsT=151;              % number of t pts
NptsX=151;              % number of x pts
t=linspace(0,tfin,NptsT);
x=linspace(-xwid,xwid,NptsX);
[T,X]=meshgrid(t,x);

figure(1)
```

Heat Equation and Fourier Transforms

```
clf
U = 50*(1 + erf(X./(sqrt(4*k*T))));   % Temperature(n)

set(gca,'FontSize',[12]);
surf(T,X,U);
shading interp
colormap(jet)
xlabel('$t$','Fontsize',12,'interpreter','latex');
ylabel('$x$','Fontsize',12,'interpreter','latex');
zlabel('$u(x,t)$','Fontsize',12,'interpreter','latex');
axis tight
view([30 12])
print -dpng heatFT1.png
print -depsc heatFT1.eps
```

Heat Equation and Fourier Transforms

```
figure(2)
clf

set(gca,'FontSize',[12]);
surf(T,X,U);
shading interp
colormap(jet)
view([0 90])                %create 2D color map of ...
    temperature
xlabel('$t$','Fontsize',12,'interpreter','latex');
ylabel('$x$','Fontsize',12,'interpreter','latex');
zlabel('$u(x,t)$','Fontsize',12,'interpreter','latex');
axis tight
colorbar
set(gca,'FontSize',[12]);
print -dpng heatFT2.png
print -depsc heatFT2.eps
```

Fourier Transforms of Derivatives

Again consider the ***Heat equation***:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad -\infty < x < \infty,$$

with **IC**, $u(x,0) = f(x)$.

Separation of variables motivated the ***Fourier transform***.

Now solve this directly with ***Fourier transform***.

Define

$$\mathcal{F}[u] = \overline{U}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{i\omega x}\,dx$$

be the ***Fourier transform*** of $u(x,t)$.

Fourier Transforms of Derivatives

Take the partial with respect to t,

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}e^{i\omega x}\,dx = \frac{\partial}{\partial t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x,t)e^{i\omega x}\,dx\right] = \frac{\partial}{\partial t}\overline{U}(\omega,t).$$

The ***spatial Fourier transform*** of a time derivative equals the time derivative of the ***Fourier transform***.

Now consider the partial with respect to x

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}e^{i\omega x}\,dx = \left.\frac{ue^{i\omega x}}{2\pi}\right|_{-\infty}^{\infty} - \frac{i\omega}{2\pi}\int_{-\infty}^{\infty}u(x,t)e^{i\omega x}dx.$$

If $\lim_{x\to\pm\infty}u(x,t) = 0$, then the endpoints vanish and

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = -i\omega\mathcal{F}[u] = -i\omega\overline{U}(\omega,t).$$

Fourier Transforms of Derivatives

Similarly, ***Fourier transforms*** of higher derivatives may be obtained:

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -i\omega\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = (-i\omega)^2\overline{U}(\omega,t) = -\omega^2\overline{U}(\omega,t).$$

In general, the ***Fourier transform*** of the n^{th} derivative of a function with respect to x equals $(-i\omega)^n$ time the ***Fourier transform*** of the function, assuming that $u(x,t) \to 0$ sufficiently fast as $x \to \pm\infty$.

From the properties of the ***Fourier transforms*** of the derivatives, the ***Fourier transform*** of the ***heat equation*** becomes:

$$\frac{\partial}{\partial t}\overline{U}(\omega,t) = -k\omega^2\overline{U}(\omega,t).$$

Fourier Transforms of Derivatives

The ***Fourier transform*** acting on the temperature function, $u(x,t)$, converts the linear partial differential equation with constant coefficients into an ordinary differential equation, since the spatial derivatives are transformed into algebraic multiples of the transform.

Since

$$\frac{\partial}{\partial t}\overline{U}(\omega,t) = -k\omega^2\overline{U}(\omega,t),$$

the solution becomes

$$\overline{U}(\omega,t) = c(\omega)e^{-k\omega^2 t},$$

where the arbitrary constant may depend on ω.

The function $c(\omega)$ comes from the **IC**, $f(x)$, so

$$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx,$$

which gives the same result as obtained by ***separation of variables***.

Convolution

The solution of the ***heat equation*** is the product of two functions of ω,

$$\overline{U}(\omega,t) = c(\omega)e^{-k\omega^2 t}.$$

Suppose $F(\omega)$ and $G(\omega)$ are ***Fourier transforms*** of $f(x)$ and $g(x)$:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx \qquad G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{i\omega x}\,dx$$

$$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega \qquad g(x) = \int_{-\infty}^{\infty} G(\omega)e^{-i\omega x}\,d\omega$$

We need to find $h(x)$ where the ***Fourier transform*** of $H(\omega)$ satisfies

$$H(\omega) = F(\omega)G(\omega).$$

Convolution

Note that

$$\begin{aligned}
h(x) &= \int_{-\infty}^{\infty} H(\omega)e^{-i\omega x}\,d\omega \;=\; \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{-i\omega x}\,d\omega, \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} g(s)e^{i\omega s}\,ds\right]e^{-i\omega x}\,d\omega, \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\left[\int_{-\infty}^{\infty} F(\omega)e^{-i\omega(x-s)}\,d\omega\right]ds, \\
h(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)f(x-s)\,ds, \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(w)g(x-w)\,dw.
\end{aligned}$$

This is the **convolution** of $f(x)$ and $g(x)$ usually denoted

$$g * f = f * g$$

Convolution and Heat Equation

For the ***heat equation***, consider the transform $\overline{U}(\omega,t)$ of the solution $u(x,t)$, where

$$\overline{U}(\omega,t) = c(\omega)e^{-k\omega^2 t}.$$

- $c(\omega)$ is the transform of the initial temperature, $f(x)$.
- $e^{-k\omega^2 t}$ is the transform of the ***fundamental solution***,

$$\sqrt{\frac{\pi}{kt}}e^{-x^2/4kt}.$$

- The **Convolution theorem** gives the solution:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\sqrt{\frac{\pi}{kt}}e^{-(x-s)^2/4kt}\,ds.$$

Convolution and Heat Equation

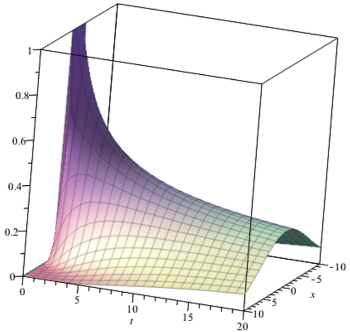
Enter the **Maple** commands for the graph of $u(x,t)$

```
u := (x,t) -> (1/(2*Pi))*(int(sqrt(Pi/t)*exp(-(1/4)*(x-s)^2/t), s = -2 .. 2));
plot3d(u(x,t), x = -10..10, t = 0.0001..20);
```

The **IC** is

$$f(x) = \begin{cases} 1, & |x| < 2, \\ 0, & |x| > 2. \end{cases}$$

This graph shows the ***diffusion*** of the heat with time.



Convolution and Heat Equation

This problem can be done in **MatLab** using its `integral` function, which uses an adaptive quadrature to solve the problem.

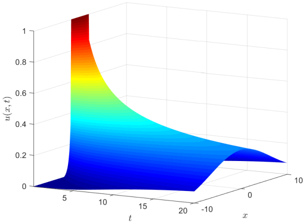
```
% Solution Heat Equation with FT
% Arbitrary f(x)

N1 = 201; N2 = 201;
tv = linspace(0.0001,20,N1);
xv = linspace(-10,10,N2);
[t1,x1] = ndgrid(tv,xv);
f = @(s,c) sqrt(pi/c(1))*exp(-(c(2)-s).^2/(4*c(1)));

for i = 1:N1
    for j = 1:N2
        c = [t1(i,j),x1(i,j)];
        U(i,j) = ...
            (1/(2*pi))*integral(@(s)f(s,c),-2,2);
    end
end
```

Convolution and Heat Equation

```
set(gca,'FontSize',[12]);
surf(t1,x1,U);
shading interp
colormap(jet)
xlabel('$t$','Fontsize',12,'interpreter','latex');
ylabel('$x$','Fontsize',12,'interpreter','latex');
zlabel('$u(x,t)$','Fontsize',12,'interpreter','latex');
axis tight
view([30 12])
```



Fourier Transforms for PDEs

The ***Fourier Transform*** technique for solving PDEs is as follows:

1. ***Fourier Transform*** the PDE in one of the variables, often x.
2. Solve the ODE in the other variable, often t.
3. Apply the ICs, determining the initial ***Fourier Transform***.
4. Use the **convolution theorem** to obtain the solution.

If the IC is only defined on a finite interval, then often **Maple** can manage the integral and produce a 3D plot.

Parseval's Identity

Since $h(x)$ is the inverse of the ***Fourier Transform*** of $F(\omega)G(\omega)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)f(x-s)\,ds = \int_{-\infty}^{\infty} G(\omega)F(\omega)e^{-\omega x}\,d\omega.$$

Since this holds for all x, it holds for $x = 0$, so

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)f(-s)\,ds = \int_{-\infty}^{\infty} G(\omega)F(\omega)\,d\omega.$$

Take $g^*(x) = f(-x)$ to be the complex conjugate, then

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(-s)e^{-i\omega s}\,ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g^*(x)e^{-i\omega x}\,dx = G^*(\omega). \end{aligned}$$

Parseval's Identity

Parseval's Identity:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)g^*(x)\,dx = \int_{-\infty}^{\infty} G(\omega)G^*(\omega)\,d\omega,$$

or equivalently,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |g(x)|^2\,dx = \int_{-\infty}^{\infty} |G(\omega)|^2\,d\omega.$$

Energy is often proportional to $|g(x)|^2$, so

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |g(x)|^2\,dx$$

is the ***total energy***.

The quantity $|G(\omega)|^2$ represents the energy per unit wave number, which is the ***spectral energy density***.

The ***Fourier Transform***, $G(\omega)$, of a function $g(x)$ is a complex quantity whose magnitude squared is the ***spectral energy density*** (or amount of energy per unit wave number).