Math 531 - Partial Differential Equations

PDEs - Higher Dimensions
Vibrating Circular Membrane

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Outline

Vibrating Circular Membrane

Vibrating Circular Membrane: The PDE satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right).$$

BC: Homogeneous Dirichlet BC:

$$u(a, \theta, t) = 0,$$

Implicit BCs: Periodic in θ (2 BCs) and Bounded

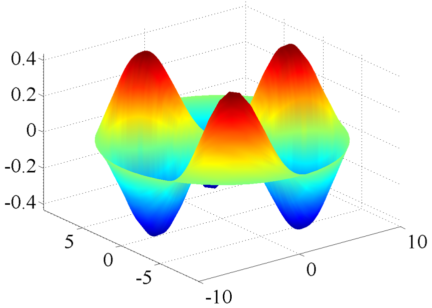
IC: Specify initial position:

$$u(r, \theta, 0) = \alpha(r, \theta),$$

Specify initial velocity:

$$u_t(r, \theta, 0) = \beta(r, \theta).$$

Solve with **Separation of Variables**.



Vibrating Circular Membrane - Separation

Consider the **Vibrating Circular Membrane** equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right).$$

Assume **separation of variables** with $u(r, \theta, t) = h(t)\phi(r)g(\theta)$, then the **PDE** becomes:

$$h''\phi g = c^2 \left(\frac{hg}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \frac{1}{r^2} h\phi g'' \right).$$

Extracting the t-dependent part of the equation gives:

$$\frac{h''}{c^2 h} = \frac{1}{r\phi} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \frac{1}{r^2 g} g'' = -\lambda.$$

Vibrating Circular Membrane - Separation

The time-dependent ODE is:

$$h'' + c^2\lambda h = 0.$$

The spatial equation can be separated:

$$\frac{g''}{g} = -\frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \lambda r^2 = -\mu.$$

The θ-dependent part satisfies the **implicit periodic BCs**, so

$$g'' + \mu g = 0, \qquad g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi).$$

The r-dependent part has an **boundedness BC** at $r = 0$ and satisfies:

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + (\lambda r^2 - \mu)\phi = 0, \qquad \phi(a) = 0.$$

Vibrating Circular Membrane - Sturm-Liouville

Two Sturm-Liouville problems for $g(\theta)$ and $\phi(r)$.

The 1^{st} **Sturm-Liouville problem** in θ is:

$$g'' + \mu g = 0, \qquad g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi).$$

This has been solved before and has ***eigenvalues***:

$$\mu_m = m^2, \qquad m = 0, 1, 2, ...$$

with corresponding ***eigenfunctions***:

$$g_0(\theta) = a_0 \qquad \text{and} \qquad g_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta).$$

Vibrating Circular Membrane - Sturm-Liouville

The 2^{nd} **Sturm-Liouville problem** in r is:

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0,$$

with the **BCs**

$$\phi(a) = 0 \qquad \text{and} \qquad |\phi(0)| \quad \text{bounded.}$$

This is a ***singular SL problem*** with $p(r) = r$, $\sigma(r) = r$, and $q(r) = \frac{m^2}{r}$.

1. The **BC** at $r = 0$ is not the correct form.
2. $p(r)$ and $\sigma(r)$ are **zero** at $r = 0$, hence not positive.
3. $q(r) \to \infty$ as $r \to 0$, so is not continuous at $r = 0$

Vibrating Circular Membrane - Sturm-Liouville

The **singular Sturm-Liouville problem**:

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)+\left(\lambda r-\frac{m^2}{r}\right)\phi=0, \qquad \phi(a)=0 \quad \text{and} \quad |\phi(0)| \text{ bounded.}$$

still has the properties of the **regular Sturm-Liouville** problem.

Significantly,

1. There are infinitely many eigenvalues, λ_{nm}, for $m=0,1,2,...$ and $n=1,2,...$ with $\lambda_{nm}>0$.
2. The eigenvalues are unbounded for each m as $n\to\infty$.
3. There are corresponding ***eigenfunctions***, $\phi_{nm}(r)$, for each λ_{nm}.
4. For each fixed m, the ***eigenfunctions*** are ***orthogonal*** with respect to the weighting function $\sigma=r$, so

$$\int_0^a \phi_{nm}(r)\phi_{km}(r)r\,dr=0, \qquad n\neq k.$$

Bessel's Differential Equation

We can rewrite the **singular Sturm-Liouville problem** as

$$r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + (\lambda r^2 - m^2)\phi = 0.$$

Make the change of variables $z = \sqrt{\lambda}r$, then

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - m^2)\phi = 0.$$

This equation has a ***regular singular point*** at $z = 0$, so can be solved by the **Method of Frobenius**, where we try solutions of the form:

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^{r+n}, \qquad \phi'(z) = \sum_{n=0}^{\infty}(r+n)a_n z^{r+n-1},$$

$$\phi''(z) = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n z^{r+n-2}.$$

Bessel's Differential Equation

When the power series, $\phi(z) = \sum_{n=0}^{\infty} a_n z^{r+n}$, is substituted into

$$z^2 \frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - m^2)\phi = 0,$$

we obtain:

$$\begin{aligned} \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n z^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n z^{r+n} & \\ -m^2 \sum_{n=0}^{\infty} a_n z^{r+n} + \sum_{n=0}^{\infty} a_n z^{r+n+2} &= 0. \end{aligned}$$

For $n = 0$, we find that

$$a_0(r^2 - m^2)z^r = 0,$$

which gives the ***indicial equation*** and shows that $r = \pm m$.

Bessel's Differential Equation

Suppose $m = 0$, so $r_{1,2} = 0$. Shifting the index on the last term, we find the series above becomes:

$$\sum_{n=0}^{\infty} n(n-1)a_n z^n + \sum_{n=0}^{\infty} n a_n z^n + \sum_{n=2}^{\infty} a_{n-2} z^n = 0.$$

or

$$\sum_{n=0}^{\infty} n^2 a_n z^n + \sum_{n=2}^{\infty} a_{n-2} z^n = 0.$$

From this we obtain that a_0 is arbitrary and $a_1 = 0$.

Also, we find the ***recurrence relation***:

$$a_n = -\frac{a_{n-2}}{n^2}.$$

It follows that

$$a_2 = -\frac{a_0}{2^2}, \qquad a_4 = \frac{a_0}{2^2 4^4}, \quad \dots, \quad a_{2k} = \frac{(-1)^k a_0}{2^{2k}(k!)^2}.$$

Bessel's Differential Equation

With $a_0 = 1$, the series solution gives the ***Bessel function of the first kind of order zero***:

$$J_0(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k}(k!)^2}, \qquad z > 0.$$

By the ***Method of Frobenius***, since the value of $r = 0$ is a repeated root, the second solution has the form

$$Y_0(z) = cJ_0(z)\ln(z) + \sum_{n=0}^{\infty} b_n z^n.$$

With some work, it can be shown that ***Bessel function of the second kind of order zero*** is

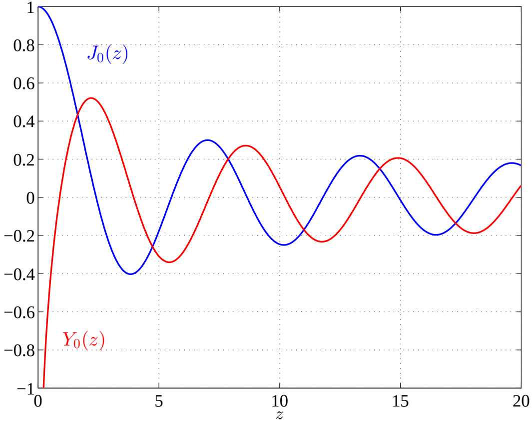
$$Y_0(z) = J_0(z)\ln(z) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} H_k z^{2k}}{2^{2k}(k!)^2},$$

where

$$H_k = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2k}.$$

Bessel's $J_0(z)$ and $Y_0(z)$

Below shows a graph of the ***Zeroth order Bessel functions of the first and second kind***. Note the many zero crossings separated by approximately π.



Bessel's $J_0(z)$ and $Y_0(z)$

MatLab code to graph Bessel functions.

```
% Bessel functions J_0(z) and Y_0(z)

z = linspace(0,20,500);

j0 = besselj(0,z);
y0 = bessely(0,z);

plot(z,j0,'b-','LineWidth',1.5);
hold on
plot(z,y0,'r-','LineWidth',1.5);
```

There is a hyperlink to **Maple code for solving Bessel's equation**.

Bessel's Equation - Asymptotic Properties

Bessel's Equation of Order m is

$$z^2 \frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - m^2)\phi = 0,$$

which has the general solution:

$$\phi(z) = c_1 J_m(z) + c_2 Y_m(z).$$

$J_m(z)$ is ***Bessel's function of the first kind of order m***.
$Y_m(z)$ is ***Bessel's function of the second kind of order m***.

Asymptotically, as $z \to 0$, $J_m(z)$ is bounded and $Y_m(z)$ is unbounded.

$$J_m(z) \sim \begin{cases} 1, & m = 0, \\ \frac{1}{2^m m!} z^m, & m > 0, \end{cases}$$

and

$$Y_m(z) \sim \begin{cases} \frac{2}{\pi}\ln(z), & m = 0, \\ -\frac{2^m (m-1)!}{\pi} z^{-m}, & m > 0. \end{cases}$$

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Bessel's Equation - Identities

There are many useful ***identities***, which have been found for Bessel functions. Below is a small list of some important ones:

1. $$\frac{d}{dx}\left(x^{-\mu}J_\mu(x)\right) = -x^{-\mu}J_{\mu+1}(x).$$

2. $$\frac{d}{dx}\left(x^{\mu}J_\mu(x)\right) = x^{\mu}J_{\mu-1}(x).$$

3. $$\int x^{\mu}J_\mu(x)x\,dx = x^{\mu}J_{\mu-1}(x).$$

EV Problem with Bessel's Equation

Our ***singular Sturm Liouville problem*** was given by

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0,$$

with boundary conditions

$$\phi(a) = 0 \qquad \text{and} \qquad |\phi(0)| \quad \text{bounded.}$$

The change of variables $z = \sqrt{\lambda}r$ converts this to ***Bessel's equation***:

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - m^2)\phi = 0.$$

Thus, the solution to the ***Sturm-Liouville problem*** is

$$\phi(r) = c_1 J_m(\sqrt{\lambda}r) + c_2 Y_m(\sqrt{\lambda}r).$$

The boundedness at $r = 0$ implies that $c_2 = 0$, so

$$\phi(r) = c_1 J_m(\sqrt{\lambda}r).$$

EV Problem with Bessel's Equation

The boundary condition $\phi(a) = 0$ means that our ***eigenvalues*** satisfy the equation:

$$J_m(\sqrt{\lambda}a) = 0.$$

Since $J_m(z)$ has infinitely many zeroes, Let z_{mn} designate the n^{th} zero of $J_m(z)$, then the ***eigenvalues*** are

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2.$$

with corresponding ***eigenfunctions***

$$\phi_{mn}(r) = J_m(z_{mn}r/a), \qquad m = 0, 1, 2, ... \quad n = 1, 2, ...$$

Numerically, we find that:

$$z_{01} \approx 2.40483, \quad z_{02} \approx 5.52008, \quad z_{03} \approx 8.65373,$$

which are approximately π apart.

EV Problem with Bessel's Equation

Recall that the ***Sturm-Liouville problem*** was

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0, \qquad \phi(a) = 0,$$

which has ***eigenvalues*** and ***eigenfunctions***;

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2, \qquad \phi_{mn}(r) = J_m(z_{mn}r/a), \qquad m = 0, 1, 2, ... \quad n = 1, 2, ...,$$

where z_{mn} is the n^{th} zero satisfying $J_m(z_{mn}) = 0$.

Since this is a ***Sturm-Liouville problem***, we have the following ***orthogonality*** condition:

$$\int_0^a J_m(\sqrt{\lambda_{mp}}r)J_m(\sqrt{\lambda_{mq}}r)r\,dr = 0, \quad p \neq q.$$

Fourier-Bessel Series

Fourier-Bessel Series: The ***eigenfunctions*** from ***Bessel's equation*** form a ***complete set***.

Take any ***piecewise smooth*** function, $\alpha(r)$, then

$$\alpha(r) \sim \sum_{n=1}^{\infty} a_n J_m(\sqrt{\lambda_{mn}}r),$$

which from the ***orthogonality*** gives the Fourier coefficients:

$$a_n = \frac{\int_0^a \alpha(r) J_m(\sqrt{\lambda_{mn}}r) r\, dr}{\int_0^a J_m^2(\sqrt{\lambda_{mn}}r) r\, dr}.$$

Return to Vibrating Membrane

Vibrating Circular Membrane: The PDE satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \quad \theta \in (-\pi, \pi], \quad r \in [0, a],$$

with **BC**: $u(a, \theta, t) = 0.$
Implicit **BCs** are

$$u(r, -\pi, t) = u(r, \pi, t), \qquad \frac{\partial u}{\partial r}(r, -\pi, t) = \frac{\partial u}{\partial r}(r, \pi, t),$$

and $|u(0, \theta, t)|$ bounded.

IC: Specify initial position, and for simplicity let it start at rest:

$$u(r, \theta, 0) = \alpha(r, \theta) \quad \text{and} \quad \frac{\partial u}{\partial t}(r, \theta, 0) = 0.$$

Return to Vibrating Membrane

Separating Variables: $u(r, \theta, t) = h(t)\phi(r)g(\theta)$, which gave the two **Sturm-Liouville problems**:

1^{st} **SL problem** in θ:

$$g'' + \mu g = 0, \qquad \text{with} \quad g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi).$$

This had ***eigenvalues*** and associated ***eigenfunctions***:

$$\mu_m = m^2, \qquad g_0(\theta) = a_0, \quad g_m(\theta) = a_n \cos(m\theta) + b_n \sin(m\theta), \quad m = 0, 1, 2, ...$$

2^{nd} **SL problem** in r:

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0, \qquad \phi(a) = 0, \quad |\phi(0)| < \infty,$$

which has ***eigenvalues*** and ***eigenfunctions***;

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2, \qquad \phi_{mn}(r) = J_m(z_{mn}r/a), \qquad m = 0, 1, 2, ... \quad n = 1, 2, ...,$$

where z_{mn} is the n^{th} zero satisfying $J_m(z_{mn}) = 0$.

Return to Vibrating Membrane

From before, $\lambda_{mn} > 0$, so the solution of the t-equation:

$$h'' + c^2\lambda_{mn}h = 0,$$

satisfies:

$$h(t) = c_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right) + d_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right).$$

The simplifying assumption that $u_t(r,\theta,0) = 0$, allows us to omit any term with $\sin\left(c\sqrt{\lambda_{mn}}t\right)$.

The **superposition principle** with our product solution gives:

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=1}^{\infty} A_{0n}J_0(\sqrt{\lambda_{0n}}r)\cos\left(c\sqrt{\lambda_{0n}}t\right) \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right)J_m(\sqrt{\lambda_{mn}}r)\cos\left(c\sqrt{\lambda_{mn}}t\right). \end{aligned}$$

Return to Vibrating Membrane

From the **IC** $u(r,\theta,0) = \alpha(r,\theta)$, we have

$$\alpha(r,\theta) = \sum_{n=1}^{\infty} A_{0n} J_0(\sqrt{\lambda_{0n}}r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right) J_m(\sqrt{\lambda_{mn}}r).$$

This produces a standard **Fourier series** in θ and a **Fourier-Bessel series** in r.

Orthogonality gives the coefficients:

$$A_{0n} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta) J_0(\sqrt{\lambda_{0n}}r) r\, dr\, d\theta}{2\pi\int_0^a J_0^2(\sqrt{\lambda_{0n}}r) r\, dr},$$

$$A_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\cos(m\theta) J_m(\sqrt{\lambda_{mn}}r) r\, dr\, d\theta}{\pi\int_0^a J_m^2(\sqrt{\lambda_{mn}}r) r\, dr},$$

$$B_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\sin(m\theta) J_m(\sqrt{\lambda_{mn}}r) r\, dr\, d\theta}{\pi\int_0^a J_m^2(\sqrt{\lambda_{mn}}r) r\, dr}.$$

Return to Vibrating Membrane

Easier notation:

$$\alpha(r,\theta) = \sum_{\lambda} A_\lambda \phi_\lambda(r,\theta),$$

where

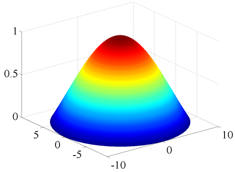
$$A_\lambda = \frac{\iint_R \alpha(r,\theta)\phi_\lambda(r,\theta)dA}{\iint_R \phi_\lambda^2(r,\theta)dA},$$

with $dA = r\,dr\,d\theta$.

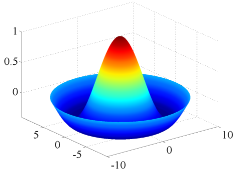
Vibrating Membrane - Fundamental Modes

Vibrating Membrane - Fundamental Modes: $m = 0$

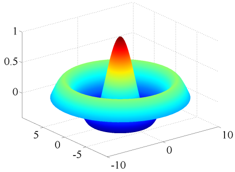
$$J_0(\sqrt{\lambda_{0n}}\,r)$$



$n = 1$



$n = 2$

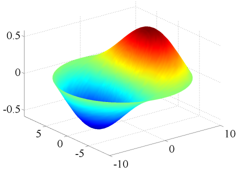


$n = 3$

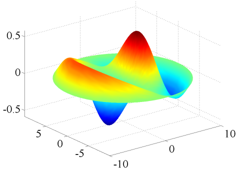
Vibrating Membrane - Fundamental Modes

Vibrating Membrane - Fundamental Modes: $m = 1$

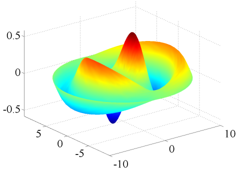
$$J_1(\sqrt{\lambda_{1n}}r)\cos(\theta)$$



$n = 1$



$n = 2$

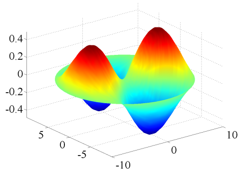


$n = 3$

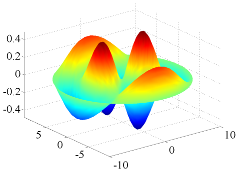
Vibrating Membrane - Fundamental Modes

Vibrating Membrane - Fundamental Modes: $m = 2$

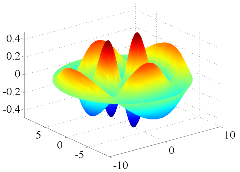
$$J_2(\sqrt{\lambda_{2n}}r)\cos(2\theta)$$



$n = 1$



$n = 2$

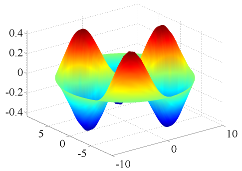


$n = 3$

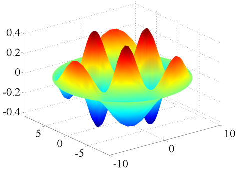
Vibrating Membrane - Fundamental Modes

Vibrating Membrane - Fundamental Modes: $m = 3$

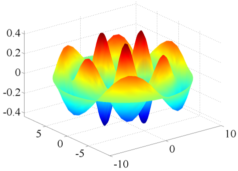
$$J_3(\sqrt{\lambda_{3n}}r)\cos(3\theta)$$



$n = 1$



$n = 2$



$n = 3$

Circularly Symmetric Case

Consider the vibrating membrane, where the region is circularly symmetric, $u = u(r,t)$:

PDE: $$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),$$

BCs: $$u(a,t) = 0, \qquad (\text{and} \quad |u(0,t)| < \infty,)$$

ICs: $$u(r,0) = \alpha(r), \qquad \frac{\partial u}{\partial t}(r,0) = \beta(r).$$

Separation of Variables: Let $u(r,t) = \phi(r)h(t)$, then

$$\phi h'' = \frac{c^2 h}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) \qquad \text{or} \qquad \frac{h''}{c^2 h} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda.$$

Time-dependent equation: This gives:

$$h'' + c^2\lambda h = 0.$$

Circularly Symmetric Case

Sturm-Liouville Problem: The spatial BVP is:

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0, \qquad \phi(a) = 0 \quad \text{and} \quad |\phi(0)| < \infty.$$

This is **Bessel's equation of Order Zero,** $m = 0$, so

$$\phi(r) = c_1 J_0\left(\sqrt{\lambda}r\right) + c_2 Y_0\left(\sqrt{\lambda}r\right),$$

which by boundedness of the solution at $r = 0$ gives $c_2 = 0$.

The **eigenvalues** satisfy λ_n, such that

$$J_0\left(\sqrt{\lambda_n}a\right) = 0,$$

with corresponding **eigenfunctions**:

$$\phi_n(r) = J_0\left(\sqrt{\lambda_n}r\right) = 0.$$

Circularly Symmetric Case

The solution of the ***time-dependent problem*** is:

$$h_n(t) = a_n \cos\left(c\sqrt{\lambda_n}t\right) + b_n \sin\left(c\sqrt{\lambda_n}t\right).$$

The **superposition principle** gives:

$$u(r,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(c\sqrt{\lambda_n}t\right) + b_n \sin\left(c\sqrt{\lambda_n}t\right)\right) J_0\left(\sqrt{\lambda_n}r\right).$$

The **initial position** gives:

$$u(r,0) = \alpha(r) = \sum_{n=1}^{\infty} a_n J_0\left(\sqrt{\lambda_n}r\right),$$

where

$$a_n = \frac{\int_0^a \alpha(r) J_0\left(\sqrt{\lambda_n}r\right) r\,dr}{\int_0^a J_0^2\left(\sqrt{\lambda_n}r\right) r\,dr}.$$

Circularly Symmetric Case

The **initial velocity** gives:

$$u_t(r,0) = \beta(r) = \sum_{n=1}^{\infty} b_n c\sqrt{\lambda_n} J_0\left(\sqrt{\lambda_n} r\right),$$

where

$$b_n = \frac{\int_0^a \beta(r) J_0\left(\sqrt{\lambda_n} r\right) r\, dr}{c\sqrt{\lambda_n}\int_0^a J_0^2\left(\sqrt{\lambda_n} r\right) r\, dr}.$$