1. a. A nutrient controlled experiment on the American Wheat Weevil, *Rhizopertha dominica*, gives a best fitting logistic growth model of the form

$$P_{n+1} = F(P_n) = P_n + 1.14P_n\left(1 - \frac{P_n}{370}\right), \qquad P_0 = 3,$$

where n is in weeks. Find all equilibria and compute the derivative of $F(P)$. Determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function.

b. An alternate model that is popular in ecology is Ricker's model with its best fitting model for the experiment above given by:

$$P_{n+1} = R(P_n) = 2.36\,P_n e^{-P_n/430}, \qquad P_0 = 3.$$

Find the derivative of the updating function $R(p)$ and determine where it has a maximum (the maximum growth of the population). Find any asymptotes and sketch a graph of the updating function.

c. Find all equilibria and determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function.

2. a. Data for the population (per cc) of a microorganism is found to reasonably fit the logistic growth model given by the discrete dynamical model:

$$P_{n+1} = F(P_n) = 2.54P_n - 0.01P_n^2, \qquad P_0 = 35,$$

where n is in days. Find all equilibria and compute the derivative of $F(P)$. Determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function.

b. A Beverton-Holt model also fits this microorganism's population (per cc)

$$P_{n+1} = B(P_n) = \frac{4.1P_n}{1 + 0.02P_n}, \qquad P_0 = 35,$$

where again n is in days. Find the derivative of the updating function. Find the P and B-intercepts. Determine any maxima or minima, including both the P and B coordinates. Find any asymptotes, then sketch a graph of this updating function.

c. Find all equilibria and determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function.

3. Evaluate the following integrals:

a. $\displaystyle\int \left(6\cos(3x) - \frac{2}{x^3}\right) dx,$

b. $\displaystyle\int \left(x^2 + 4\,x - 5\right)^3 (x+2)dx,$

c. $\displaystyle\int \left(\frac{3}{x^2} + 3\,\cos(3x-2)\right) dx,$

d. $\displaystyle\int \left(2x\,e^{-x^2} - 4\,x\right)dx.$

e. $\displaystyle\int \left(4e^{-2x} + \frac{3}{\sqrt{x}}\right) dx,$

f. $\displaystyle\int \left(\frac{7}{x} + 8\sin^3(4x)\cos(4x)\right) dx.$

4. Evaluate the following definite integrals:

a. $\displaystyle\int_0^{2\pi} (\cos(t/4) + t)\, dt,$

b. $\displaystyle\int_1^2 \left(\frac{6x}{x^2+1} + \frac{2}{x}\right) dx.$

c. $\displaystyle\int_0^5 x\sqrt{25 - x^2}\, dx,$

d. $\displaystyle\int_1^4 \left(\frac{\ln(x)}{x} + \frac{3}{\sqrt{x}}\right) dx.$

5. Solve the following initial value problems:

a. $\dfrac{dy}{dt} = 1 + e^{-t}, \quad y(0) = 3,$

b. $\dfrac{dy}{dt} = 2 - \dfrac{4}{t}, \quad y(1) = 5,$

c. $\dfrac{dy}{dt} = \dfrac{3t^2}{2y}, \quad y(0) = 4,$

d. $\dfrac{dy}{dt} = 2 - 0.02\,y, \quad y(0) = 5,$

e. $\dfrac{dy}{dt} = \dfrac{2\,ty}{t^2+1}, \quad y(0) = 3,$

f. $\dfrac{dy}{dt} = (2 - 0.2t)y, \quad y(0) = 10,$

g. $\dfrac{dy}{dt} = 4 - 2\sin(2(t-3)), \quad y(3) = 5,$

h. $\dfrac{dy}{dt} = e^{t-y}, \quad y(0) = 6,$

6. The population of the United States was about 50.2 million in 1880 and 62.9 million in 1890. Let 1880 be represented by $P(0)$ and assume that its population is growing according to the Malthusian growth law,

$$\frac{dP(t)}{dt} = rP(t),$$

where t is in years.

a. Use the data above to find the growth rate r, then solve the differential equation above. Determine how long until the U. S. population doubled from its 1880 level according to this model.

b. Predict the population in the year 1900. The actual population was about 76.0 million. What is the error between the model and the actual census data?

7. Radioactive iodine, ^{131}I, is used to treat a number of thyroid problems, including hyperthyroidism, Grave's disease, and thyroid cancer. The first two of these problems require only 3-12 mCi (millicuries) of ^{131}I, so can be treated using an outpatient procedure. However, the cancer treatment often uses doses of 30 mCi and requires a special isolation room where everything must be disposed of as radioactive waste, including urine, feces, bedsheets, and eating utensils, since the ^{131}I is contained in all the bodily fluids (though concentrated in the thyroid).

a. Radioactive iodine, ^{131}I, has a half-life of 8 days and satisfies the differential equation

$$\frac{dR}{dt} = -kR, \quad R(0) = R_0.$$

Assume that $R_0 = 30$ mCi, then find the decay constant k (to 3 significant figures). If the patient is isolated for 3 days, then how many millicuries of ^{131}I remain either in the patient or with the waste products collected.

b. How long does it take for the original 30 mCi dose to decay to only 5 mCi? (Note that most of the ^{131}I will have passed out through the fluids and not be in the body by this time.)

c. Find the maximum cumulative exposure over the first 20 days by integrating this radium sample:

$$\int_0^{20} R(t)\, dt.$$

(Note that the patient will have received only a fraction of this exposure, since much of the radioactive iodine will have passed out through fluid excretion.)

8. a. White lead is a pigment found in oil paints and can be used to detect art forgeries. In the absence of radium-226, lead-210 undergoes standard radioactive decay,

$$\frac{dP}{dt} = -kP.$$

Suppose that a sample from a painting has 10 disintegrations per minute in 1970 and then shows 8.5 disintegrations per minute in 1975. Find the half-life of lead-210 and give the value of k.

b. When there are impurities caused by radium-226 (which has a very long half-life), the differential equation for radioactive decay is modified to

$$\frac{dP}{dt} = -kP + r,$$

where $r = 0.25$ is source input from the radium-226 and k is from Part a. Solve this differential equation and determine the limit of P (disintegrations per minute of lead-210) as $t \to \infty$.

9. You are attending a conference, and the talks are going past the coffee break time. You really need a cup of tea (not liking coffee) to keep awake for the next set of talks. The refreshments are in a room that has a constant temperature of 21°C, and you find that the hot water is only 85°C. Five minutes later, the hot water is only 81°C.

a. Assume that the container of water satisfies Newton's law of cooling. ($H' = -k(H - T_e)$, where T_e is the environmental temperature.) If it was placed out when the talks were supposed to end with boiling water (water at 100°C), then how many minutes beyond the scheduled time did the talks go? (Hint: If $H(t)$ is the temperature, then use $H(0) = 85$ and $H(5) = 81$ to find the cooling constant k in Newton's law of cooling, then find when $H(t) = 100$.)

b. If tea needs water that is at least 93°C to give you enough caffeine for the next set of talks, then how long after the scheduled end of the talks can you wait?

10. a. An initially clean lake ($c(0) = 0$) concentrates pollution from an incoming stream because of evaporative loss of water of 200 m^3/day. The well-mixed lake has a stream flowing in at a rate of $f_1 = 2200$ m^3/day with a pesticide concentration of of $Q = 10$ ppb. The lake maintains a constant volume of $V = 10^6$ m^3 by having a stream leaving with a flow of $f_2 = 2000$ m^3/day. You are given

that the differential equation describing the concentration of pesticide in the lake is given by

$$c' = \frac{1}{V}(f_1 Q - f_2 c).$$

Solve this differential equation.

b. Determine how long until the lake has a concentration of 5 ppb of pesticide. Also, find the limiting concentration of pesticide. Sketch a graph of the solution.

11. A kangaroo can leap vertically 8 ft. Determine equations describing the velocity and height of the kangeroo as functions of time using this assumption on the maximum height it can achieve, that is find $v(t)$ and $h(t)$ including numerical values of all constants in these formulae. How long is the kangeroo in the air and what is the animal's initial upward velocity? (Use the acceleration due to gravity as 32 ft/sec^2.)

12. It has been shown that the radial spread of a disease in an orchard satisfies the differential equation

$$\frac{dT}{dt} = k\sqrt{T},$$

where T is the number of diseased trees and t is in years. Suppose that initially there is a single diseased tree ($T(0) = 1$) and that 4 years later $T(4) = 25$. Solve this differential equation. Find the value of k, then determine how many trees are infected after 10 years.

13 a. The population of Japan was 116.8 million in 1980, while in 1990, it was 123.5 million. Let 1980 be $t = 0$ (giving the initial population) and assume that Japan's population satisfies the Malthusian growth law given by

$$\frac{dJ}{dt} = rJ, \qquad J(0) = J_0,$$

where r is the growth rate, J_0 is the initial population, and t is time in years. Solve this growth model for Japan, giving the growth rate r from these data to **4 significant figures**. How long does it take for Japan's population to double according to this model?

b. The population of Bangladesh was 88.1 million in 1980, while in 1990, it was 110.1 million. Again, let 1980 be $t = 0$ and assume that Bangladesh's population satisfies a Malthusian growth law. Write a differential equation describing the population of Bangladesh, $B(t)$, solve this equation, and use the model to predict Bangladesh's population in 2000. (Give the growth rate to **4 significant figures**.)

c. Find when the population of Japan is equal to the population of Bangladesh according to the models above.

14. An older woman is quite ill, and her daughter finds that she has been running a temperature of 39°C. Over the night, the woman passes away in her sleep, and the daughter discovers her death at 7 AM. At this time the body is found to be 35°C. Two hours later the body temperature is 33.5°C. The woman's bedroom maintained a temperature of 25°C. If the body satisfies Newton's law of cooling,

$$\frac{dH(t)}{dt} = -k(H(t) - T_e),$$

where T_e is the temperature of the bedroom, t is in hours, H is the temperature in °C, and k is the coefficient of heat transfer to be determined (to **4 significant figures**) for this woman. Determine when the woman died (using normal time, hours and minutes).

15. a. A new experimental drug is being tested for its efficacy on treatment of tumors. The patient (who has none of this drug in her body to begin with) has 10 liters of blood. The drug enters the blood intravenously at a rate of 1 liter/day at a concentration of 0.2 μg/liter. The body mixes this drug well throughout the circulatory system, then it filters out the drug through the kidneys at a rate proportional to the concentration of the drug in the blood with a flow rate of 1 liter/day. Set up the differential equation that describes the concentration, $c(t)$, of this new drug and solve it.

b. Find when the drug concentration reaches a concentration of 0.1 μg/liter in the body, so that the tumor responds.

c. Assume the same scenario as given above for the drug entering and leaving the patient's body. If it is found that the body metabolizes (removes) 0.05 μg/day of the drug, then write the new differential equation that describes the concentration of this drug in the body. What is the limiting concentration of the drug in the body for this model?

16. a. A colony of bacteria grows according to the Malthusian growth model

$$\frac{dB}{dt} = 0.01\,B, \qquad B(0) = 1000,$$

where t is in min. Solve this differential equation and determine how long it takes for this population to double.

b. Because it takes a short time to adjust to the new medium, a better model is given by

$$\frac{dB}{dt} = 0.01(1 - e^{-t})\,B, \qquad B(0) = 1000.$$

Solve this differential equation.

c. Compare the populations predicted at $t = 5$ and 60 min.

17. a. Consider the Malthusian growth model for a particular animal that has recently colonized some region

$$\frac{dP}{dt} = 0.2\,P, \qquad P(0) = 100,$$

where t is in years. Solve this differential equation and determine how long it takes for this population to double.

b. Because of habitat encroachment, this animal is losing its range for expansion. This results in a growth rate that is time dependent. Suppose that the population satisfies the modified Malthusian growth model

$$\frac{dP}{dt} = (0.2 - 0.02\,t)\,P, \qquad P(0) = 100.$$

Solve this differential equation.

c. Find the maximum of this population and what year this occurs. Also, determine when the population returns to 100. Sketch a graph for this population.

18. a. Consider the growth of a population of cells in a declining medium. If the population growth depends on the absorption of the medium through the cell surface and the medium is decaying exponentially, then a differential equation for this population is given by

$$\frac{dP}{dt} = 0.3e^{-0.01t}P^{2/3}, \qquad P(0) = 1000,$$

where the initial population is 1000 and t is in hours. Solve this differential equation.

b. Find how long it takes for this population to double. What happens to this population for very large time (*i.e.*, find any horizontal asymptotes)? Sketch a graph for this population.

19. a. An initially clean pond that contains 10,000 m^3 of water maintains a constant volume. The stream flows in at a rate of 200 m^3/day with 10 μg/m^3 of phosphate (from fertilizer) in the stream. The pond is well-mixed, and a stream flows out at the same rate. Write a differential equation that describes the concentration of phosphate $c(t)$ in the lake, then solve this equation.

b. Algae grows well on phosphate. The rate of growth of algae is proportional to the concentration of phosphate and the population of algae $A(t)$ to the 2/3 power,

$$\frac{dA}{dt} = 0.05\,c(t)A^{2/3}, \qquad A(0) = 1000.$$

Find the population of algae at any time t.

20. There is a tremendous controversy about swordfish. It is a very popular in sport fishing and top ocean predator. There is a boycott of this type of fish because the average harvest size has dropped well below the size of sexual maturity. It is also one of the earliest fish that was recognized to have a buildup of mercury. Sadly, the current swordfish that are being served are too small and young to even have a chance to buildup toxic levels of mercury with the average catch size being only 40 kg, which is less than 3 years old.

a. Swordfish can get extremely large, exceeding 1000 kg. However, it grows slowly (and if not overfished can live a long time), so the weight of a swordfish satisfies the following differential equation:

$$\frac{dw}{dt} = 0.015(1000 - w), \qquad w(0) = 0,$$

where w is the weight in kg and t is the time in years. Solve this differential equation. Use the results to determine how long it takes to produce a mature 70 kg swordfish.

b. The mercury (Hg) accumulates as the swordfish grows and is not removed. Assume that the intake of Hg is proportional to the weight of the swordfish, so satisfies the differential equation

$$\frac{dH}{dt} = kw(t), \qquad H(0) = 0,$$

with $k = 0.01$ (mg of Hg/kg-yr) and H being the mg of Hg in a swordfish. Solve this differential equation. Find the amount of Hg in swordfish that are 3 and 20 years old.

c. If the Hg is uniformly spread in the swordfish, then the concentration of Hg, $c(t)$ (in μg/g), would be given by the formula

$$c(t) = H(t)/w(t).$$

Find the weight of swordfish, $w(t)$, and concentration of Hg, $c(t)$, at times $t = 3$ and 20 years. (Note that officials get concerned when Hg level reaches 0.1 μg/g.)

21. Studies of Lake Apopka in Florida show that the alligators there have been exposed to high levels of various estrogen-simulating pesticides, like DDT (or its breakdown DDE), dieldrin, and toxaphenes. Apparently, this exposure has resulted in dramatic decrease in the size of the male alligator penises, which in turn finally spurred our Congress into action. (They were unconcerned when it was shown to have adverse effects on female animal populations.) The levels of these estrogenic pesticides in this lake far exceed the safe levels established by the EPA. The effect on penis development is clearly related to the amount of exposure of the embryo, which in turn reflects the amount in the body of the female alligators.

a. The growth (weight) of a female alligator can be approximated by the following differential equation:

$$\frac{dw}{dt} = 0.2(80 - w), \qquad w(0) = 0,$$

where w is the weight in kg and t is the time in years. Solve this differential equation. Use the results to determine how long it takes to produce a mature 40 kg alligator.

b. The pesticides accumulate (especially in the fatty tissues) as the alligator grows and is not removed. Assume that the intake of pesticides is proportional to the weight of the alligator, so satisfies the differential equation

$$\frac{dP}{dt} = kw(t), \qquad P(0) = 0,$$

with $k = 600$ (μg/kg-yr) and P being the μg of pesticides in an alligator at Lake Apopka. Solve this differential equation. Find the amount of pesticide in an alligator that is 5 years old.

c. The concentration, $c(t)$ (in μg/g or ppm), is found by computing

$$c(t) = \frac{P(t)}{1000w(t)}.$$

Find the concentration in a 5 year old alligator. (Note that officials get concerned when pesticide levels reach 0.1 ppm in an animal.)

22. A new pesticide is introduced to a particular region, where a stream becomes contaminated with the pesticide and flows into a 10^6 m^3 lake. The stream flows at a rate of 4000 m^3/day with a concentration of 15 ng/m^3.

a. Assuming that the lake is well-mixed and maintains a constant volume, then the differential equation describing the concentration of this new pesticide in the lake is given by the differential equation:

$$\frac{dc}{dt} = 0.004(15 - c), \qquad \text{with} \qquad c(0) = 0.$$

Solve this differential equation and find the limiting concentration in the lake.

b. More realistically, the flow of the stream is seasonal and fluctuates over the year by about 40%. The lake still maintains an almost constant volume, but a better model for the concentration of the pesticide in the lake is given by:

$$\frac{dc}{dt} = -0.001(4 - \cos(0.0172t))(c - 15), \qquad \text{with} \qquad c(0) = 0.$$

Solve this differential equation. Be sure to show your steps for this calculation, including the separation of variables and the solution of each of the integrals.

23. a. The decay of a particular fruit with total mass, M, satisfies the following differential equation:

$$\frac{dM}{dt} = -k\, M^{3/4}, \qquad M(0) = 16 \text{ g},$$

where t is in days. It is found that after 10 days only 1 g remains of the fruit, so $M(10) = 1$. Solve this differential equation. Find the value of k. Determine when the fruit completely vanishes ($M(t_f) = 0$).

b. A special culture of bacteria is added to the decaying fruit, and it is found that the decaying fruit satisfies the differential equation:

$$\frac{dM}{dt} = -0.8\, e^{-0.02t}\, M^{3/4}, \qquad M(0) = 16 \text{ g},$$

Solve this differential equation. Find the length of time for this fruit to completely vanish.

24. a. A population study is conducted on a new colony of invasive insects. The study finds that initially there are 60 of the insects in 1 m^2, $P(0) = 60$. Two weeks later a survey finds 80 of the insects in 1 m^2, $P(2) = 80$. Assume that this insect population is growing according to a Malthusian growth law:

$$\frac{dP}{dt} = r\, P,$$

where t is in weeks. Solve this differential equation, find the growth constant r, and determine how long it takes for the total population to double.

b. It is found that a predator is adapting to the new invasive insect and is learning to control this pest. A survey after four weeks finds the population has only increased to 90, $P(4) = 90$. The result is a declining growth rate and a better model for the population is given by the differential equation:

$$\frac{dP}{dt} = (a - bt)P.$$

Solve this differential equation. Use the data at $t = 0$, 2, and 4 weeks to find the constants a and b. Determine the time for this population to reach its maximum and what the maximum population is predicted to be.

25. Consider the curves $y = 3 - x$ and $y = 6 + x - x^2$.

a. Find all the x and y-intercepts for both curves. Determine the slope of the line and the vertex of the parabola. Find the points of intersection, then sketch the graph of these curves.

b. Set up and solve the integral that determines the area between the two curves.

26. Two researchers analyze six years of population data for a particular animal that is given in the table below.

Year	0	1	2	3	4	5	6
Pop	54	73	85	89	86	75	53

a. The first researcher fits the data with the quadratic equation

$$P(t) = 54 + 24t - 4t^2.$$

Find the maximum population using this approximation to the data and when it occurs. (Show how you compute this maximum.)

b. The second researcher fits the data with the curve

$$Q(t) = 54 + 34\sin\left(\frac{\pi}{6}t\right).$$

Sketch a graph of this curve and give where the maximum population occurs.

c. Find the average populations using Parts a. and b. by computing the definite integrals

$$P_{ave} = \frac{1}{6}\int_0^6 P(t)dt \qquad \text{and} \qquad Q_{ave} = \frac{1}{6}\int_0^6 Q(t)dt.$$