Spring 2016 Math 531 Take-Home Exam 2 **Due Mon April 25, 2016 by 5 PM**

I, ________________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

For all of the problems below, perform all integrations that can be readily be done. Use orthogonality to eliminate any zero coefficients. State clearly your reference for any short-cutted solutions to Sturm-Liouville problems.

1. a. Find the eigenvalues and eigenfunctions for the Sturm-Liouville problem

$$\begin{aligned} u'' + \lambda u &= 0, \qquad & 0 < x < \pi, \\ u(0) &= 0, \qquad & 2u(\pi) + u'(\pi) = 0. \end{aligned}$$

b. Use the eigenfunctions from Part a to represent the function

$$f(x) = \begin{cases} 3, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x < \pi \end{cases}.$$

and find the generalized Fourier coefficients.

c. What does the Fourier series converge to at $x = 2$? at $x = \frac{\pi}{2}$? at $x = -1$? Does this Fourier series produce a periodic extension for all x? Explain.

d. Use the computer to find the numerical values of the first 50 eigenvalues. Graphically, show $f(x)$ and the approximation using 50 terms in the Fourier series. What is the error between your 50 term Fourier series and the value of $f(x)$ at $x = 0.05$, $x = 1$, $x = 2$, and $x = 1.65$. Find the maximum error between the 50 term approximation and the actual function. (The maximum error is for the Gibb's phenomenon and not the obvious error caused by the jump discontinuity, which pointwise approaches 3.)

2. Find the steady-state temperature in a cube, which satisfies:

$$\nabla^2 u(x, y, z) = 0, \qquad 0 < x < 2, \quad 0 < y < 2, \quad 0 < z < 2.$$

The cube is insulated on the faces with $x = 0$ and $y = 2$. The cube is kept at 0°C on the faces with $x = 2$ and $z = 0$ and kept at T_0 when $y = 0$. Finally, it satisfies Newton's law of cooling on the other face ($z = 2$) with

$$-k\frac{\partial u(x, y, 2)}{\partial z} = hu(x, y, 2).$$

3. A can of beer at room temperature (20°C) is almost submersed in ice water (0°C).

a. Find the steady-state temperature of the beer assuming it satisfies Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}=0, \qquad 0<r<1, \quad 0<z<4,$$

with the boundary conditions:

$$u(1,z)=0, \quad u(r,0)=0, \quad u(r,4)=20.$$

You can assume there is infinite ice. For **extra-credit**, assume that you pour this beer into a glass (beer becomes well-mixed, so takes the average steady-state temperature), don't assume any heat transfer from the glass, then use 50 terms in your solution to determine what is the average temperature of the beer.

b. In this part of the problem, we want to know the time evolution of the cooling of the beer. The can of beer satisfies the heat equation:

$$\frac{\partial u}{\partial t}=k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}\right), \qquad 0<r<1, \quad 0<z<4, \quad t>0,$$

with the boundary conditions:

$$u(1,z,t)=0, \quad u(r,0,t)=0, \quad u(r,4,t)=20,$$

and initial condition:

$$u(r,z,0)=20.$$

Find the temperature of the beer, $u(r,z,t)$ for all $t>0$.

4. Consider heat conduction in a sphere given by:

$$\frac{\partial u}{\partial t}=\frac{k}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right), \qquad 0<\rho<a, \quad t>0,$$

with the boundary and initial conditions:

$$u(a,t)=0, \quad u(\rho,0)=T_0.$$

Solve this equation noting any other boundary conditions you might need to apply. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships. (Hint: You might want to try the change of variables given by $u(\rho,t)=v(\rho,t)/\rho$.)