1. Consider the integral with the substitution $u = 2x^2 - 3$, so $du = 4x\,dx$. It follows that

$$\int x\sqrt{2x^2-3}\,dx = \frac{1}{4}\int (2x^2-3)^{\frac{1}{2}}(4x)dx = \frac{1}{4}\int u^{\frac{1}{2}}du$$
$$= \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}}\right)+C = \frac{1}{6}(2x^2-3)^{\frac{3}{2}}+C.$$

2. Consider the integral with the substitution $u = 2x - 6$, so $du = 2\,dx$. It follows that

$$\int \frac{4}{2x-6}\,dx = 2\int \frac{2}{2x-6}dx = 2\int u^{-1}du$$
$$= 2\ln|u|+C = 2\ln|2x-6|+C.$$

3. Consider the integral with the substitution $u = x^2 + 4x - 5$, so $du = (2x+4)dx$. It follows that

$$\int \frac{x+2}{(x^2+4x-5)^3}\,dx = \frac{1}{2}\int \frac{2x+4}{(x^2+4x-5)^3}\,dx = \frac{1}{2}\int u^{-3}du$$
$$= \frac{1}{2}\left(\frac{u^{-2}}{-2}\right)+C = -\frac{1}{4(x^2+4x-5)^2}+C.$$

4. Consider the integral with the substitution $u = x^2 + 4$, so $du = 2x\,dx$. It follows that

$$\int x\sin(x^2+4)\,dx = \frac{1}{2}\int \sin(x^2+4)\,(2x)dx = \frac{1}{2}\int \sin(u)du$$
$$= -\frac{1}{2}\cos(u)+C = \frac{-\cos(x^2+4)}{2}+C.$$

5. Consider the integral with the substitution $u = x^2 - 2x$, so $du = (2x-2)\,dx$. It follows that

$$\int \frac{(x-1)}{e^{x^2-2x}}\,dx = \frac{1}{2}\int (2x-2)e^{-(x^2-2x)}dx = \frac{1}{2}\int e^{-u}du$$
$$= -\frac{1}{2}e^{-u}+C = -\frac{1}{2}e^{-x^2+2x}+C.$$

6. Consider the integral with the substitution $u = \cos(2x) + 6$, so $du = -2\sin(2x)\,dx$. It follows that

$$\int \frac{5\sin(2x)}{(\cos(2x)+6)}\,dx = \frac{5}{-2}\int \frac{-2\sin(2x)}{(\cos(2x)+6)}\,dx = -\frac{5}{2}\int \frac{1}{u}du$$
$$= -\frac{5}{2}\ln|u| + C = -\frac{5}{2}\ln|\cos(2x)+6| + C.$$

7. Consider the integral with the substitution $u = (x+7)^{\frac{1}{2}}$, so $du = \frac{1}{2(x+7)^{\frac{1}{2}}}\,dx$. It follows that

$$\int \frac{e^{\sqrt{x+7}}}{\sqrt{x+7}}\,dx = 2\int \frac{e^{(x+7)^{\frac{1}{2}}}}{2(x+7)^{\frac{1}{2}}}\,dx = 2\int e^u du$$
$$= 2e^u + C = 2e^{\sqrt{x+7}} + C.$$

8. Consider the integral with the substitution $u = 1 - x^4$, so $du = -4x^3\,dx$. It follows that

$$\int \frac{x^3}{\sqrt{1-x^4}}\,dx = -\frac{1}{4}\int (1-x^4)^{-\frac{1}{2}}(-4x^3)\,dx = -\frac{1}{4}\int u^{-\frac{1}{2}}du$$
$$= -\frac{2}{4}u^{\frac{1}{2}} + C = -\frac{1}{2}(1-x^4)^{\frac{1}{2}} + C.$$

9. This differential equation has only a function of t on the right, so it can be solved by simply integrating the right hand side.

$$y(t) = \int \left(t\sqrt{t^2+1}\right)dt = \frac{1}{2}\int (t^2+1)^{\frac{1}{2}}2t\,dt.$$

With the substitution $u = t^2 + 1$, so $du = 2t\,dt$,

$$y(t) = \frac{1}{2}\int u^{\frac{1}{2}}\ du = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C = \frac{1}{3}(t^2+1)^{\frac{3}{2}} + C.$$

With the initial condition, $y(0) = 5 = \frac{1}{3}(1)^{\frac{3}{2}} + C$ or $C = \frac{14}{3}$. Thus,

$$y(t) = \frac{1}{3}(t^2+1)^{\frac{3}{2}} + \frac{14}{3}.$$

10. This differential equation has only a function of t on the right, so it can be solved by simply integrating the right hand side.

$$y(t) = \int \left(t\sin(t^2-4)\right)dt = \frac{1}{2}\int \left(\sin(t^2-4)\right)2t\ dt.$$

With the substitution $u = t^2 - 4$, so $du = 2t\,dt$,

$$y(t) = \frac{1}{2}\int (\sin(u))\,du = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(t^2-4) + C.$$

With the initial condition, $y(2) = 3 = -\frac{1}{2}\cos(2^2-4) + C$ or $C = \frac{7}{2}$. Thus,

$$y(t) = \frac{7}{2} - \frac{1}{2}\cos(t^2-4).$$

11. The differential equation, $\frac{dy}{dt} = \frac{ty}{\sqrt{t^2-1}}$, is a separable differential equation. Thus,

$$\int \frac{dy}{y} = \int \frac{t}{\sqrt{t^2-1}}dt = \frac{1}{2}\int \frac{2t}{\sqrt{t^2-1}}dt.$$

With the substitution $u = t^2 - 1$, so $du = 2t\,dt$,

$$\ln|y| = \frac{1}{2}\int u^{-\frac{1}{2}}du = u^{\frac{1}{2}} + C = \sqrt{(t^2-1)} + C.$$

Solving for y gives

$$y(t) = Ae^{\sqrt{(t^2-1)}}, \qquad \text{with} \quad A = e^C.$$

With the initial condition, $y(1) = 4 = A$. Thus,

$$y(t) = 4e^{\sqrt{t^2-1}}.$$

12. The differential equation, $\frac{dy}{dt} = 0.1t(4-y) = -0.1t(y-4)$, is a separable differential equation. Thus,

$$\int \frac{dy}{y-4} = -0.1\int t dt.$$

With the substitution $u = y - 4$, so $du = dy$,

$$\int \frac{du}{u} = \ln|u| = \ln|y(t) - 4| = -\frac{0.1t^2}{2} + C.$$

Solving for y gives

$$|y(t) - 4| = Ae^{-0.05t^2}, \qquad \text{with} \quad A = e^C.$$

With the initial condition, $y(0) = 10$, so $|10 - 4| = 6 = A$. Thus,

$$y(t) = 4 + 6e^{-0.05t^2}.$$

13. The differential equation, $t\frac{dy}{dt} = 2\,(\ln(t))^4$, when divided by t has only a function of t on the right hand side. Thus,

$$y(t) = 2\int (\ln(t))^4\,\frac{1}{t}dt.$$

With the substitution $u = \ln(t)$, so $du = \frac{dt}{t}$,

$$y(t) = 2\int u^4 du = 2\frac{u^5}{5} + C = \frac{2}{5}(\ln(t))^5 + C.$$

With the initial condition, $y(1) = 3 = C$, so

$$y(t) = \frac{2}{5}(\ln(t))^5 + 3.$$

14. The differential equation, $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{50{,}000}\right)$, is a logistic differential equation, which can be separated and solved following the technique shown in the lecture. This begins with the fractional decomposition:

$$\frac{1}{P\left(\frac{P}{50{,}000} - 1\right)} = \frac{1}{50,000}\left(\frac{1}{\frac{P}{50{,}000} - 1}\right) - \frac{1}{P}.$$

Thus, separation of variables gives

$$\frac{1}{50,000}\int\left(\frac{1}{\frac{P}{50{,}000} - 1}\right) dP - \int\frac{dP}{P} = -0.2\int dt = -0.2t + C.$$

With the substitution, $u = \frac{P}{50{,}000} - 1$, so $du = \frac{dP}{50{,}000}$. If follows that:

$$\int\frac{du}{u} - \ln|P(t)| = \ln\left|\frac{P(t)}{50,000} - 1\right| - \ln|P(t)| = -0.2t + C.$$

By properties of logarithms,

$$\ln\left|\frac{\frac{P(t)}{50{,}000} - 1}{P(t)}\right| = -0.2t + C \qquad \text{or} \qquad \left|\frac{1}{50,000} - \frac{1}{P(t)}\right| = Ae^{-0.2t}.$$

From the initial condition, $P(0) = 1000$, so

$$\left|\frac{1}{50,000} - \frac{1}{1000}\right| = A = \frac{49}{50,000}.$$

Since $P(t) < 50,000$, we have

$$\frac{1}{P(t)} - \frac{1}{50,000} = \frac{49}{50,000}e^{-0.2t} \qquad \text{or} \qquad \frac{1}{P(t)} = \frac{1 + 49e^{-0.2t}}{50,000},$$

so

$$P(t) = \frac{50,000}{1 + 49e^{-0.2t}}.$$

When the population doubles $P(t) = 2000$, so

$$2000 = \frac{50,000}{1 + 49e^{-0.2t}} \qquad \text{or} \qquad 1 + 49e^{-0.2t} = \frac{50,000}{2000} = 25.$$

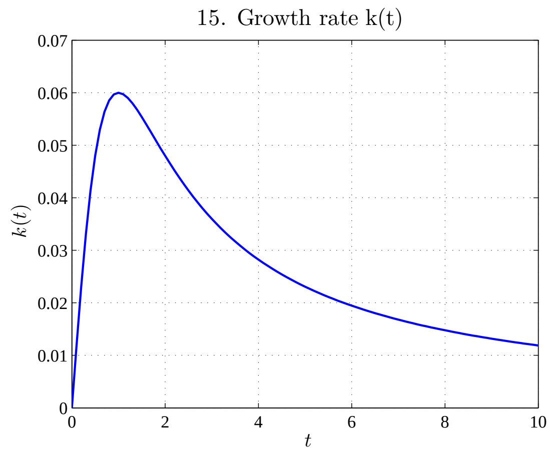
It follows that $e^{0.2t} = \frac{49}{24}$ or $t = 5\ln\left(\frac{49}{24}\right) \approx 3.5688$ hr.

Since the carrying capacity is 50,000, half the carrying capacity is reached when $P(t) = 25,000 = \frac{50,000}{1+49e^{-0.2t}}$ or $1 + 49e^{-0.2t} = 2$. Thus, $t = 5\ln(49) \approx 19.459$ hr.

15. a. Consider $k(t) = \frac{0.12t}{t^2+1}$, so we differentiate and obtain:

$$k'(t) = 0.12\frac{(t^2+1)\cdot 1 - t(2t)}{(t^2+1)^2} = 0.12\frac{1-t^2}{(t^2+1)^2}$$

This function has critical points at $t_{cr} = \pm 1$. It is easy to see there is a **maximum** at $t_{max} = 1$. The maximum value is $k(t_{max}) = 0.06$. The graph of $k(t)$ is shown below.



b. The differential equation, $\frac{dV}{dt} = \frac{0.12t}{t^2+1}V^{\frac{2}{3}}$, is separable. It can be written in the form:

$$\int V^{-\frac{2}{3}} dV = 0.12\int (t^2+1)^{-1} t\ dt = 0.06\int (t^2+1)^{-1} 2t\ dt.$$

We make the substitution $u = t^2 + 1$, so $du = 2t\ dt$, so

$$3V^{\frac{1}{3}} = 0.06\int u^{-1} du = 0.06\ln|u| + C = 0.06\ln(t^2+1) + C.$$

Solving for $V(t)$,

$$V(t) = \left(0.02\ln(t^2+1) + C/3\right)^3.$$

The initial condition is $V(0) = 1$, so $C = 3$, which gives the solution

$$V(t) = \left(1 + 0.02\ln(t^2+1)\right)^3.$$

c. For the cell to double in volume, $V(t) = 2 = (1 + 0.02\ln(t^2+1))^3$. Thus, $\sqrt[3]{2} = 1 + 0.02\ln(t^2+1)$ or $\ln(t^2+1) = 50\left(\sqrt[3]{2} - 1\right)$. It follows that

$$t^2 = e^{50(\sqrt[3]{2}-1)} - 1 \approx 440669 \qquad \text{or} \qquad t \approx 664 \text{ min.}$$