Calculus for the Life Sciences I

Lecture Notes – Product Rule

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Spring 2013

Outline

Tumor Growth) 1

Cancer and Tumor Growth: Mathematical Role

- Image Processing

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1

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- Image Processing
- Calculating therapeutic doses

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Cancer and Tumor Growth: Mathematical Role

- Image Processing
- Calculating therapeutic doses
- Epidemiology of cancer in a population
- Growth of tumors

Tumor Growth

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- The center of the tumor largely consists of dead cells, called the **necrotic center** of the tumor
- The tumor grows outward in roughly a spherical shell shape

Gompertz Growth Model 1

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  - However, can be shown that

$$\lim_{N\to 0} G(N) = 0$$

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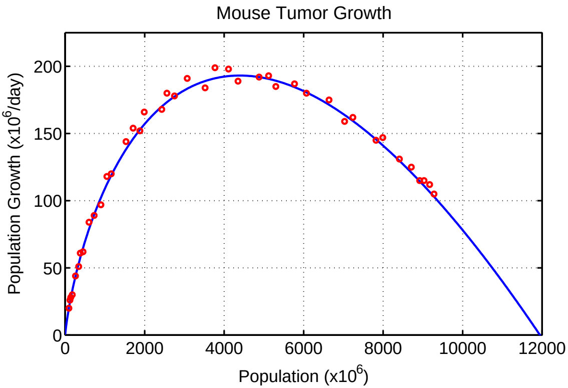
- They studied the C3H Mouse Mammary tumor
- Tritiated thymidine was used to measure the cell cycles
- This gave the growth rate for these tumors

Gompertz Growth Model

Mouse Tumor Growth and Gompertz Model: The best fit to the Gompertz Model is

$$G(N) = N(0.4126 - 0.0439\ln(N))$$



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  - Most cancer therapies attack growing cells
  - Treatment has its maximum effect when maximum growth is occurring

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This is the unique equilibrium of the **Gompertz Model** or its **carrying capacity**

For the mouse tumor data above

$$N_e = e^{0.4126/0.0439} = e^{9.399} = 12,072,$$

which matches the P-intercept on the graph

Maximum Growth from Gompertz Model

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- Our techniques of Calculus can find the maximum – set the derivative equal to zero
- Finding the derivative of $G(N)$ presents a new problem in differentiation
- We need the **product rule for differentiation** to differentiate $G(N)$

Product Rule

Product Rule: Let $f(x)$ and $g(x)$ be differentiable functions. The product rule for finding the derivative of the product of these two functions is given by:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\frac{dg(x)}{dx} + \frac{df(x)}{dx}g(x)$$

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In words, this says that the **derivative of the product of two functions** is the **first function times the derivative of the second function plus the second function times the derivative of the first function**

Product Rule - Example

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From the **product rule**

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$$\begin{aligned} f'(x) &= f_1(x)f_2'(x) + f_1'(x)f_2(x) \\ &= x^2(3x^2) + (2x)x^3 = 5x^4 \end{aligned}$$

Example – Product Rule

Example: Consider the function

$$f(x) = (x^3 - 2x)(x^2 + 5)$$

Find the derivative of $f(x)$

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Solution: From the **product rule**

$$f'(x) = (x^3 - 2x)(2x) + (x^2 + 5)(3x^2 - 2)$$

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$$\begin{aligned} f'(x) &= (x^3 - 2x)(2x) + (x^2 + 5)(3x^2 - 2) \\ &= 2\,x^4 - 4\,x^2 + 3\,x^4 - 2\,x^2 + 15\,x^2 - 10 \end{aligned}$$

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$$g'(x) = (x^2 + 4)\frac{1}{x} + (\ln(x))(2x)$$

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Solution: From the **product rule**

$$\begin{aligned} g'(x) &= (x^2+4)\frac{1}{x} + (\ln(x))(2x) \\ g'(x) &= x + \frac{4}{x} + 2x\,\ln(x) \end{aligned}$$

Maximum Growth for the Gompertz Tumor Growth Model 1

Maximum Growth for the Gompertz Tumor Growth Model:

Apply the Product Rule to the Gompertz Growth function

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Applied to the Gompertz model for the mouse mammary tumor, then the maximum occurs at the population

$$N_{max} = e^{(9.399-1)} = 4,441(\times 10^6)$$

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Applied to the Gompertz model for the mouse mammary tumor, then the maximum occurs at the population

$$N_{max} = e^{(9.399-1)} = 4,441(\times 10^6)$$

Substituted into the Gompertz growth function, the maximum growth of mouse mammary tumor cells is

$$G(N_{max}) = 4441(0.4126 - 0.0439 \ln(4441)) = 195.0(\times 10^6/\text{day})$$

Ricker Function

Example – Ricker Function: Consider the Ricker function

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- Find points of inflection
- Sketch the graph

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By the **product rule**, the derivative is

$$\frac{dR}{dx} = 5x(-0.1\, e^{-0.1x}) + 5\, e^{-0.1x} = 5\, e^{-0.1x}(1 - 0.1\, x)$$

Ricker Function 2

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Since the exponential is never zero, the only **critical point** satisfies

$$1 - 0.1\,x = 0 \quad \text{or} \quad x = 10$$

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$$1 - 0.1\, x = 0 \quad \text{or} \quad x = 10$$

There is a maximum at

$$(10, 50\, e^{-1}) \quad \text{or} \quad (10, 18.4)$$

Ricker Function 3

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$$\frac{d^2R}{dx^2} = 5\,e^{-0.1x}(-0.1)+5(-0.1)e^{-0.1x}(1-0.1x) = 0.5\,e^{-0.1x}(0.1\,x-2)$$

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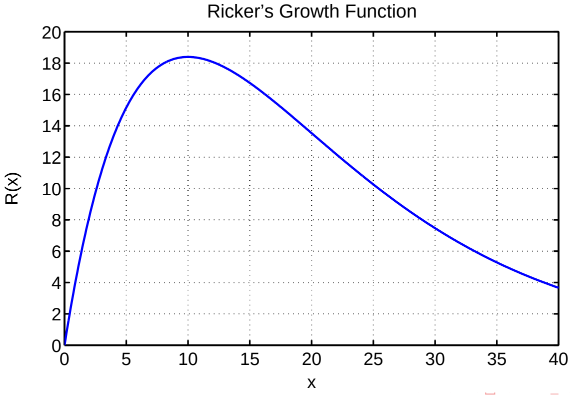
- The point of inflection occurs at $x = 20$

$$(20, 100\,e^{-2}) \quad \text{or} \quad (20, 13.5)$$

Ricker Function

Solution (cont): Graph of the Ricker function

$$R(x) = 5x\,e^{0.1x}$$



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- Sketch the graph of $f(x)$ for $0 < x \leq 2$

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- The x-intercept is found by solving $f(x) = 0$, which gives $x = 1$

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Solution (cont): For $f(x) = x\,\ln(x)$ by the product rule the derivative is

$$f'(x) = x\left(\frac{1}{x}\right) + \ln(x) = 1 + \ln(x)$$

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$$\ln(x_c) = -1 \quad \text{or} \quad x_c = e^{-1} \approx 0.3679$$

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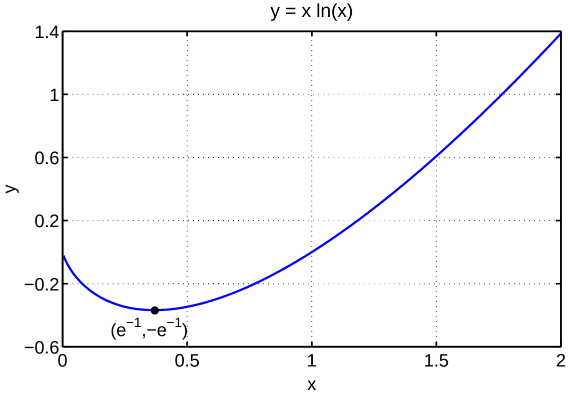
- The function value at the critical point is

$$f(e^{-1}) = -e^{-1} \approx -0.3679$$

- There is a **minimum** on the graph at $(e^{-1}, -e^{-1})$

Example – Graphing

Solution (cont): The graph of $f(x) = x\,\ln(x)$ is



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- It can be shown

$$\lim_{x \to -\infty} f(x) = 0$$

Example – Graphing 2

Solution: For $f(x) = (2-x)e^x$

- Since $f(0) = 2$, the y-intercept is $(0,2)$
- Since the exponential function is never zero, the x-intercept is $(2,0)$
- It can be shown

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  - $f(x)$ goes to 0, so there is a **horizontal asymptote** to the left at $y = 0$

Example – Graphing

3

Solution (cont): For $f(x) = (2-x)e^x$ by the product rule the derivative is

$$f'(x) = (2-x)e^x + (-1)e^x =$$

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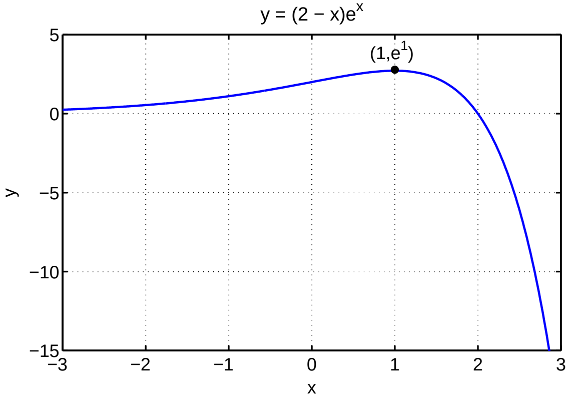
- There is a **maximum** on the graph at $(1, e^1)$

Example – Graphing

Solution (cont): The graph of $f(x) = (2 - x)e^x$ is



Example – Growth of Tumor 1

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where W is the weight of the tumor in mg

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Example: Suppose the growth of a tumor satisfies Gompertz growth function

$$G(W) = W(0.5 - 0.05\ln(W)),$$

where W is the weight of the tumor in mg

- Find the equilibrium weight of the tumor
- Find the maximum growth rate for this tumor
- Sketch the graph of $G(W)$

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Example – Growth of Tumor 2

Solution: The **equilibrium** is found by solving $G(W)$ equal to zero

$$
\begin{aligned}
G(W) &= W(0.5 - 0.05\ln(W)) = 0 \\
0.5 - 0.05\ln(W) &= 0 \\
\ln(W) &= 10 \\
W &= e^{10} = 22,026 \text{ mg}
\end{aligned}
$$

Example – Growth of Tumor 3

Solution cont): The **maximum growth** is found by setting the derivative $G'(W) = 0$

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Example – Growth of Tumor 3

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$$\begin{aligned} G'(W) &= W\left(-\frac{0.05}{W}\right) + (0.5 - 0.05\,\ln(W)) \\ G'(W) &= 0.45 - 0.05\,\ln(W) \;=\; 0 \end{aligned}$$

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Example – Growth of Tumor

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The maximum growth rate

$$G(8,103) = 8,103(0.5 - 0.05\ln(8,103)) = 405.2 \text{ mg/day}$$

Example – Growth of Tumor 4

Solution (cont): The graph of

$$G(W) = W(0.5 - 0.05\ \ln(W)),$$