

1. a. From the transition matrix, the distribution of tagged frogs after 1, 2, 5, and 10 days are given by:

$$\begin{pmatrix} f_1(1) \\ f_2(1) \\ f_3(1) \\ f_4(1) \end{pmatrix} = \begin{pmatrix} 42 \\ 7 \\ 34 \\ 17 \end{pmatrix} \quad \begin{pmatrix} f_1(2) \\ f_2(2) \\ f_3(2) \\ f_4(2) \end{pmatrix} = \begin{pmatrix} 27.94 \\ 15.97 \\ 37.54 \\ 18.55 \end{pmatrix}$$

$$\begin{pmatrix} f_1(5) \\ f_2(5) \\ f_3(5) \\ f_4(5) \end{pmatrix} = \begin{pmatrix} 23.17 \\ 20.66 \\ 35.64 \\ 20.53 \end{pmatrix} \quad \begin{pmatrix} f_1(10) \\ f_2(10) \\ f_3(10) \\ f_4(10) \end{pmatrix} = \begin{pmatrix} 23.06 \\ 20.69 \\ 35.47 \\ 20.78 \end{pmatrix}$$

b. The expected distribution of the frogs after a long period of time is given by

$$[23.06\%, \quad 20.69\%, \quad 35.47\%, \quad 20.78\%].$$

The most suitable habitat is clearly Region 3, while it would be hard to distinguish the least suitable habitats amongst the others, since they differ by less than 2.5% though the ranking from the values above are Region 2, Region 4, then Region 1.

2. a. From the Malthusian growth model

$$P_{n+1} = 1.03P_n, \quad P_0 = 100,$$

the population after 20 years is 180.61. The length of time for this population to double is 23.45 years.

b. A MatLab program describing a birth only model for 100 individuals with each individual having a 3% chance of producing an offspring each year is given by the following:

```
function v=pop2(p)
pt = ones(1,100);
n = sum(pt);
v = n;
for i=1:20,
    b = (rand(1,n)<=p*pt);
    nb = sum(b);
    np = ones(1,nb);
    pt = [pt,np];
    n = sum(pt);
    v = [v,n];
end
```

The program is executed by typing `pop2(0.03)` in MatLab. A typical result is given by:

[100, 105, 108, 110, 113, 119, 126, 130, 139, 141, 146, 149, 155, 162, 167, 172, 179, 188, 194, 198, 203].

c. The program above is simulated 1000 times with the following program:

```

function Mpop2(p,M)
for i=1:M,
    v = pop2(p);
    Pop10(i) = v(11);
    Pop20(i) = v(21);
end MeanPop10 = mean(Pop10)
STDPop10 = std(Pop10)
MeanPop20 = mean(Pop20)
STDPop20 = std(Pop20)

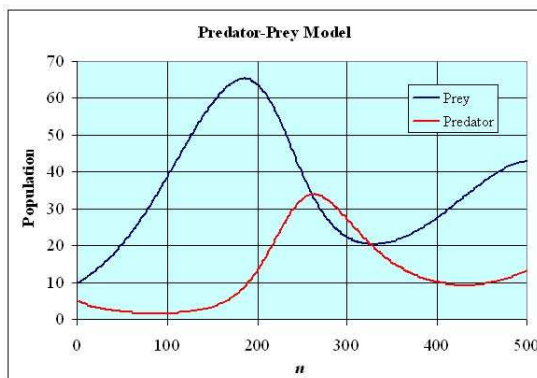
```

A typical result gives the average population at 10 as 134.50 with a standard deviation of 6.3799. A typical average population at 20 is 180.866 with a standard deviation of 11.5909. Note that this average population over 1000 simulations is very similar to the calculations in Part a.

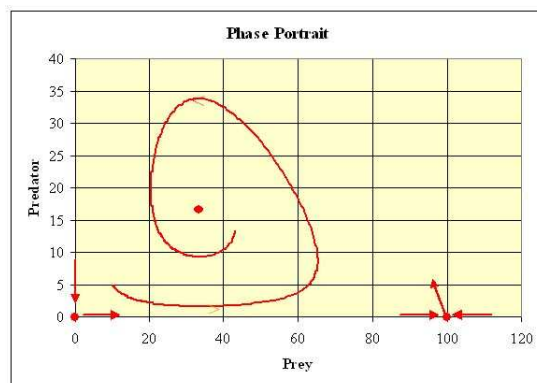
3. a. The predator-prey model

$$\begin{aligned}
 x_{n+1} &= 1.02x_n - 0.002x_n^2 - 0.004x_ny_n \\
 y_{n+1} &= 0.97y_n + 0.009x_ny_n.
 \end{aligned}$$

has x_n as the prey and y_n as the predator. In the x_n equation, the first term is the Malthusian growth, the second term is the intraspecies competition, and the last term is the loss of prey due to predation. In the y_n equation, the first term is a linear loss from lack of prey, and the second term is the gain of predators from consuming the prey. This model shows decaying oscillations towards a stable equilibrium. The strengths of the model are the simple form of the equations showing the basic interactions and the easy to interpret results with a stable equilibrium. The weaknesses are the lack of realistic factors such as age-structure, spatial distribution, and seasonal variation. Below is a graph showing a simulation of the model.



Time simulation



Phase portrait

b. There are three equilibria for this model. The first equilibrium is $(x_e, y_e) = (0, 0)$, which has the eigenvalues $\lambda_1 = 1.02$ and $\lambda_2 = 0.97$ with corresponding eigenvectors $\xi_1 = [1, 0]$ and $\xi_2 = [0, 1]$ and is a saddle node. The second equilibrium is $(x_e, y_e) = (100, 0)$, which has the eigenvalues $\lambda_1 = 0.98$ and $\lambda_2 = 1.06$ with corresponding eigenvectors $\xi_1 = [1, 0]$ and $\xi_2 = [-1, 1]$ and is a saddle node. The third equilibrium is $(x_e, y_e) = (33.33, 16.67)$, which has the eigenvalues $\lambda = 0.9967 \pm 0.01972i$ with corresponding eigenvectors $\xi = [-1.315 \mp 0.2222i, i]$ and is a stable spiral as the absolute value of the eigenvalues are $|\lambda| = 0.9967 < 1$.

c. The phase portrait is shown above. The populations will ultimately stabilize at the third coexistence equilibrium.

4. a. From the tables, we find the following information on the birth rates and survivals:

$$s_{12} = 0.3871 \quad s_{23} = 0.8848 \quad s_{34} = 0.7255 \quad b_1 = 0.7974 \quad b_2 = 1.8992 \quad b_3 = 2.8524.$$

b. This results in the Leslie model of the following form:

$$\begin{pmatrix} P_1(n+1) \\ P_2(n+1) \\ P_3(n+1) \\ P_4(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 0.797 & 1.90 & 2.85 \\ 0.387 & 0 & 0 & 0 \\ 0 & 0.885 & 0 & 0 \\ 0 & 0 & 0.7255 & 0 \end{pmatrix} \begin{pmatrix} P_1(n) \\ P_2(n) \\ P_3(n) \\ P_4(n) \end{pmatrix}.$$

The estimated population in each of the age classes for the next 5 years is:

$$\begin{pmatrix} P_1(6) \\ P_2(6) \\ P_3(6) \\ P_4(6) \end{pmatrix} = \begin{pmatrix} 514.6 \\ 164.5 \\ 128.3 \\ 78.35 \end{pmatrix} \quad \begin{pmatrix} P_1(7) \\ P_2(7) \\ P_3(7) \\ P_4(7) \end{pmatrix} = \begin{pmatrix} 598.2 \\ 199.1 \\ 145.6 \\ 93.10 \end{pmatrix} \quad \begin{pmatrix} P_1(8) \\ P_2(8) \\ P_3(8) \\ P_4(8) \end{pmatrix} = \begin{pmatrix} 700.6 \\ 231.5 \\ 176.2 \\ 105.6 \end{pmatrix}$$

$$\begin{pmatrix} P_1(9) \\ P_2(9) \\ P_3(9) \\ P_4(9) \end{pmatrix} = \begin{pmatrix} 820.3 \\ 271.1 \\ 204.9 \\ 127.9 \end{pmatrix} \quad \begin{pmatrix} P_1(10) \\ P_2(10) \\ P_3(10) \\ P_4(10) \end{pmatrix} = \begin{pmatrix} 969.8 \\ 317.5 \\ 240.0 \\ 148.6 \end{pmatrix}$$

c. The eigenvalues and eigenvectors for the Leslie matrix are:

$$\lambda_1 = 1.1735, \quad \xi_1 = \begin{pmatrix} 0.9151 \\ 0.3018 \\ 0.2276 \\ 0.1407 \end{pmatrix} \quad \lambda_{2,3} = -0.1925 \pm 0.8533i, \quad \xi_{2,3} = \begin{pmatrix} 0.8074 \\ -0.0786 \mp 0.3485i \\ -0.3264 \pm 0.1551i \\ 0.1851 \pm 0.2358i \end{pmatrix}$$

$$\lambda_4 = -0.7886, \quad \xi_4 = \begin{pmatrix} 0.7452 \\ -0.3657 \\ 0.4104 \\ -0.3775 \end{pmatrix}$$

From the dominant eigenvalue $\lambda_1 = 1.1735$, we obtain a limiting age-class distribution

$$[57.73\%, 19.04\%, 14.36\%, 8.88\%]$$

This gives an approximate annual growth of 17.4% with a doubling time of 4.33 years.

5. a. From the data on the yeast, we find the best fitting parameters to be $a = 0.070276$ and $M = 172.64$ with the initial condition $X_0 = 12.465$. The sum of square errors is 18.051. Thus, the model is given by:

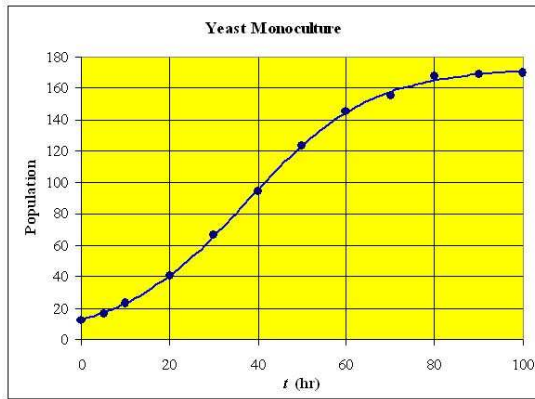
$$X_{n+1} = X_n + 0.070276X_n \left(1 - \frac{X_n}{172.64}\right).$$

Below is a graph of the model and the data.

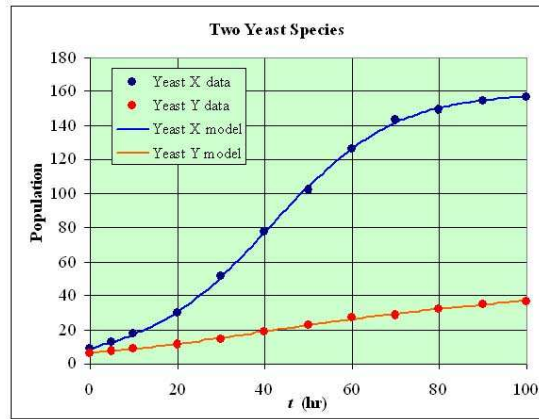
b. With the constants a, b, M , and N from the monoculture experiments, a least squares best fit to the competition data was performed, giving the parameters $a_3 = 6412.5$ and $b_3 = 9030.6$ and initial conditions $X_0 = 9.2202$ and $Y_0 = 6.4387$. The sum of square errors was 10.829. Thus, the model is given by:

$$X_{n+1} = X_n + 0.070276X_n \left(1 - \frac{X_n}{172.64}\right) - \frac{X_n Y_n}{6412.5},$$

$$Y_{n+1} = Y_n + 0.03475Y_n \left(1 - \frac{Y_n}{112.9}\right) - \frac{X_n Y_n}{9030.6}.$$



One Species



Two Species

Below is a graph of the model and the data.

c. There are four equilibria for this model. The first equilibrium is $(x_e, y_e) = (0, 0)$, which has the eigenvalues $\lambda_1 = 1.0703$ and $\lambda_2 = 1.03475$ with corresponding eigenvectors $\xi_1 = [1, 0]$ and $\xi_2 = [0, 1]$ and is an unstable node. The second equilibrium is $(x_e, y_e) = (172.64, 0)$, which has the eigenvalues $\lambda_1 = 0.9297$ and $\lambda_2 = 1.0156$ with corresponding eigenvectors $\xi_1 = [1, 0]$ and $\xi_2 = [-0.31337, 1]$ and is a saddle node. The third equilibrium is $(x_e, y_e) = (0, 112.9)$, which has the eigenvalues $\lambda_1 = 0.96525$ and $\lambda_2 = 1.0527$ with corresponding eigenvectors $\xi_1 = [0, 1]$ and $\xi_2 = [1, -0.1430]$ and is a saddle node. The fourth equilibrium is $(x_e, y_e) = (150.07, 58.91)$, which has the eigenvalues $\lambda_1 = 0.9852$ and $\lambda_2 = 0.9356$ with corresponding eigenvectors $\xi_1 = [0.4769, -0.94265]$ and $\xi_2 = [0.9902, 0.1396]$ and is a stable node.

d. The phase portrait is shown below. The populations will ultimately converge to the coexistence equilibrium.

