1. Since 1973, the British Forestry Commission has surveyed for the presence of the American gray squirrel (*Sciurus carolinensis Gmelin*) and the native red squirrel (*Sciurus vulgaris L.*). From two consecutive years of data for 10 km square regions across Great Britain, data were collected on movement of the two types of squirrels. The transition matrix for red squirrels, gray squirrels, both, or neither in that order was given by

$$T = \begin{pmatrix} 0.8797 & 0.0382 & 0.0527 & 0.0008 \\ 0.0212 & 0.8002 & 0.0041 & 0.0143 \\ 0.0981 & 0.0273 & 0.8802 & 0.0527 \\ 0.0010 & 0.1343 & 0.0630 & 0.9322 \end{pmatrix}.$$

Find the equilibrium distribution of squirrels based on this transition matrix. Does this model suggest that the invasive gray species will significantly displace the native red squirrel over long periods of time?

2. a. Consider an animal that lives five years and reproduces annually. Animals that are 0-1 years old don't reproduce and only 35% ($s_1 = 0.35$) of them survive to the next year. Animals that are 1-2 years old are inexperienced and produce on average $b_2 = 0.7$ offspring and 60% ($s_2 = 0.6$) of them survive to the next year. Animals 2-3 years old produce on average $b_3 = 1.6$ offspring and 75% ($s_3 = 0.75$) of them survive to the next year. The peak age is animals 3-4 years old, who produce $b_4 = 2.5$ offspring and 70% ($s_4 = 0.70$) of them survive to the next year. Finally, animals 4-5 years old produce $b_5 = 1.8$ offspring. Create a model using a Leslie matrix, L, of the form:

$$P_{n+1} = LP_n.$$

Find the steady-state percentage of each age group. Determine how long it takes for this population to double after it has reached its steady-state distribution.

b. Assume that a fraction of 3-5 year olds are harvested. That is, the survival rates s_3 and s_4 are reduced. If the survival rates are reduced by a fraction α, so that the survival rate of 2-3 year olds is $0.75\,\alpha$ and the survival rate of 3-4 year olds is $0.7\,\alpha$. Determine the value of α that leaves the population at a constant value. For this value of α (to at least 4 significant figures), if there are 350 mature (4-5 year olds), then determine the total population and number in each population age group. How many animals are harvested annually under these conditions?

3. A. C. Crombie [1] studied *Rhizopertha dominica*, the lesser grain borer, and *Oryzaephilus surinamensis*, the saw-tooth grain beetle, competing for the same resource.

a. From the data, a competition model can be derived, and it has the following form:

$$\begin{aligned} R_{n+1} &= 1.044R_n - 0.00013R_n^2 - 0.000024R_nO_n, \\ O_{n+1} &= 1.069O_n - 0.00016O_n^2 - 0.000014R_nO_n, \end{aligned}$$

where n is in days and R_n represents *Rhizopertha dominica* and O_n represents *Oryzaephilus surinamensis*. Simulate this model for 100 days assuming that

$$R_0 = 5 \qquad \text{and} \qquad O_0 = 1.$$

Show the time series graph of both populations.

b. Find all equilibria and find the eigenvalues at those equilibria. Discuss the stability of each of the equilibria and predict what will happen with the populations of these beetles over a long period of time, assuming the experimental conditions hold. Sketch a phase portrait of the populations (R on the horizontal axis and O on the vertical axis) showing the equilibria and indicating the eigenvectors and direction of population change relative to the equilibria.

4. a. The classic Lotka-Volterra or predator-prey model examines the interaction of a predator with its prey species. The classic model assumes that the predator eats almost exclusively the prey animal and that the prey animal suffers little loss except for predation by the primary predator. Suppose that a predator and prey interaction satisfies the equation:

$$\begin{aligned} H_{n+1} &= H_n + 0.025H_n - 0.00045H_nP_n, \\ P_{n+1} &= P_n - 0.065P_n + 0.00015H_nP_n, \end{aligned}$$

where n is in days. Determine which species is the predator and which species is the prey. Simulate this model for 100 days assuming that

$$H_0 = 200 \qquad \text{and} \qquad P_0 = 40.$$

Show the time series graph of both populations.

b. Find all equilibria and find the eigenvalues at those equilibria. Sketch a phase portrait of the populations (H on the horizontal axis and P on the vertical axis) showing the equilibria and local change in population near the equilibria.

c. Suppose that we modify the model above to include an intraspecies competition term for the prey. The new model becomes:

$$\begin{aligned} H_{n+1} &= H_n + 0.025H_n - 0.00002H_n^2 - 0.00045H_nP_n, \\ P_{n+1} &= P_n - 0.065P_n + 0.00015H_nP_n, \end{aligned}$$

where n is in days. Simulate this model for 100 days assuming that

$$H_0 = 200 \qquad \text{and} \qquad P_0 = 40.$$

Show the time series graph of both populations.

d. Find all equilibria and find the eigenvalues at those equilibria. Sketch a phase portrait of the populations (H on the horizontal axis and P on the vertical axis) showing the equilibria and local change in population near the equilibria.

5. Gonorrhea ranks high among reportable communicable diseases in the United States. Public health officials estimate that more than 2,500,000 Americans contract the disease every year. This disease is spread by sexual contact and if untreated can result in blindness, sterility, arthritis, heart failure, and possibly death. Gonorrhea has a very short incubation time (3-7 days) and does not confer immunity to those individuals who have recovered from the disease.

It often causes itching and burning for males, particularly during urination, while it is often asymptomatic in females. Thus, males tend to seek treatment more often than females.

We will assume a sexually active heterosexual population with c_1 females and c_2 males. If the number of infective females is given by x and the number of infective males is given by y, then a mathematical model that describes this disease is given by the following discrete dynamical system:

$$\begin{aligned} x_{n+1} &= x_n - a_1 x_n + b_1(c_1 - x_n)y_n), \\ y_{n+1} &= y_n - a_2 y_n + b_2(c_2 - y_n)x_n), \end{aligned}$$

where the cure rates for infective females and males are proportional to the infective populations with proportionality constants a_1 and a_2, respectively. New infective females are added to the population at a rate proportional to the number of infective males and susceptible females, $b_1(c_1 - x)y$. (A similar term adds infectives to the male population.)

a. Assume a sexually active female population of $c_1 = 1000$ and a sexually active male population of $c_2 = 1000$. Begin the initial infected populations with $x_0 = 20$ females and $y_0 = 20$ males. Since males seek treatment more frequently for this disease, we assume the treatment rates of $a_1 = 0.33$ for females and $a_2 = 0.53$ for males. Take the rate of transmission (or infective rate) for each population to be similar, say $b_1 = 0.00052$ and $b_2 = 0.00047$. Simulate the disease for 100 months (or 100 iterations), graphing the populations of infected females and males on a single graph (being sure to label which population is which). Find all equilibria for this model with these parameters. Give the eigenvalues at all equilibria and sketch a phase portrait, showing the equilibria and the local behavior near the equilibria. Discuss what is happening to this disease from your graph, including a reason why one population has a higher number of infected individuals.

b. Because AIDS is such a problem in today's society, the sexually active population is taking more precautions. This may result in a lowering of the infective rates. Let $b_1 = 0.00042$ and $b_2 = 0.00037$. Let x_0 and y_0 be the last population of infectives from your simulation in Part a. Now simulate the model for 50 months. Graph your solution and discuss how this affects the model. Find all equilibria for this model with these parameters. Give the eigenvalues at all equilibria and sketch a phase portrait, showing the equilibria and the local behavior near the equilibria.

c. Alternately, people fearing the more dangerous disease AIDS may seek medical treatment earlier. This would affect the treatment rate. Let $a_1 = 0.43$ for females and $a_2 = 0.63$. (Use the same b_1 and b_2 from Part a.) Again let x_0 and y_0 be the last population of infectives from your simulation in Part a and simulate the model for 50 months. Graph your solution and discuss how this affects the model. Find all equilibria for this model with these parameters.

d. Discuss the biological significance of your simulations and what it predicts about the disease. What are the strengths of this model for discussing public policy toward sexually transmitted diseases? What are the weaknesses that you see in the model?