1. Michael Crichton in the *Andromeda Strain* (1969) states that "A single cell of the bacterium *E. coli* would, under ideal circumstances, divide every twenty minutes... [I]t can be shown that in a single day, one cell of *E. coli* could produce a super-colony equal in size and weight to the entire planet Earth." A single *E. coli* has a volume of about 1.7 μm^3. The diameter of the Earth is 12,756 km, so assuming it is a perfect sphere, determine how long it takes for an ideally growing colony (Malthusian growth) of *E. coli* (doubling every 20 min) to equal the volume of the Earth.

2. When a monoculture of an organism is grown in a limited (but renewed) medium, then the population of that organism often follows the logistic growth model. Below is a table for the growth of a fresh water organism.

Day	0	2	3	5	7	9	11	12	15
Population(/cc)	2	5	9	27	63	109	186	211	227

a. Consider the discrete logistic growth model

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{M}\right), \qquad P(0) = P_0.$$

Use the data above to find the parameters P_0, r, and M that best fit this model. Write the least sum of square error. Graph the model and the data. What is the carrying capacity for this model?

b. Now consider the continuous version of the logistic growth model, which is given by

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right), \qquad P(0) = P_0.$$

Use the data above to find the parameters P_0, r, and M that best fit this model. Write the least sum of square error. Write the solution to this differential equation. Again, graph the model and the data. What is the carrying capacity for this model?

c. Which model fits the data best? Which model is easier to use and why? List at least one advantage of each model. List at least one problem with each model.

3. a. Suppose that a population of fish, $F(t)$ (in thousands), satisfies the logistic growth model given by

$$\frac{dF}{dt} = 0.4\,F\left(1 - \frac{F}{200}\right),$$

where t is in years. Find all equilibria for this model. Sketch a graph of the right hand side of the model, then draw the phase portrait on the F-axis. What is the stability of each of the equilibria? Determine the carrying capacity for this population of fish.

b. Assume that fishing is allowed and that 15,000 fish are harvested annually. The model becomes

$$\frac{dF}{dt} = 0.4\,F\left(1 - \frac{F}{200}\right) - 15.$$

Find all equilibria for this model. Sketch a graph of the right hand side of the model, then draw the phase portrait on the F-axis. What is the stability of each of the equilibria? Determine the carrying capacity for this population of fish. What is the threshhold number of fish needed to avoid extinction?

c. If the model for fishing with a harvesting level of h (in thousands) is given by

$$\frac{dF}{dt} = 0.4\,F\left(1 - \frac{F}{200}\right) - h,$$

then determine the maximum level of harvesting that allows the fish population exist without going to extinction.

4. An enclosed area is divided into four regions with varying habitats. One hundred tagged frogs are released into the first region. Earlier experiments found that on average the movement of frogs each day about the four regions satisfied the transition model given by

$$\begin{pmatrix} f_1(n+1) \\ f_2(n+1) \\ f_3(n+1) \\ f_4(n+1) \end{pmatrix} = \begin{pmatrix} 0.42 & 0.16 & 0.19 & 0.16 \\ 0.07 & 0.38 & 0.24 & 0.13 \\ 0.34 & 0.19 & 0.51 & 0.27 \\ 0.17 & 0.27 & 0.06 & 0.44 \end{pmatrix} \begin{pmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \\ f_4(n) \end{pmatrix}.$$

a. Give the expected distribution of the tagged frogs after 1, 2, 5, and 10 days.

b. What is the expected distribution of the frogs after a long period of time? Which of the four regions is the most suitable habitat and which is the least suitable for these frogs?

5. A mathematical model for the growth of a benign tumor that is limited by its nutrient supply has been shown to satisfy the Gompertz differential equation. The number of cancer cells (in thousands), $N(t)$, satisfies the differential equation

$$\frac{dN}{dt} = -rN\,\ln\left(\frac{N}{M}\right), \qquad N(0) = N_0,$$

where t is in weeks.

a. Find the general solution to this differential equation.

b. Suppose that measurements on the growth of a tumor give the following data:

t (weeks)	0	10	20	30	40	50
N (in thousands)	50	500	1200	1600	1800	1900

Using the solution to the Gompertz differential equation, find the nonlinear least squares best fit to the data with the parameters r, M, and N_0.

c. Find what happens to the tumor for very large time.

6. An age-structured population of birds were surveyed over 5 years. The researchers determined the number of birds in each age class for each of the 5 years and found out how many nestlings fledged from each of the different age classes each year. These birds typically live only 4 years. Assume a typical age-structured model of the form

$$\begin{pmatrix} P_1(n+1) \\ P_2(n+1) \\ P_3(n+1) \\ P_4(n+1) \end{pmatrix} = \begin{pmatrix} 0 & b_2 & b_3 & b_4 \\ s_{12} & 0 & 0 & 0 \\ 0 & s_{23} & 0 & 0 \\ 0 & 0 & s_{34} & 0 \end{pmatrix} \begin{pmatrix} P_1(n) \\ P_2(n) \\ P_3(n) \\ P_4(n) \end{pmatrix}.$$

a. The table below shows how many birds in each age class survived to the next year (and gives the total number of birds that fledged. Use the data below to compute the average values for each of the survival parameters s_{12}, s_{23}, and s_{34}.

Bird Age	Year 1	Year 2	Year 3	Year 4	Year 5
0-1	210	275	319	371	425
1-2	75	80	105	126	145
2-3	60	62	73	99	108
3-4	38	44	46	54	68

They also collected data on the success rate of nesting of each of the different age classes of birds. The table below shows the number of fledgings raised by each of the age classes over the 5 year period. (Note that these columns total to the number of 0-1 year old birds the next year. Use the data below to compute the average birth rates for each of the age classes b_2, b_3, and b_4. (One year old birds of this species don't nest.)

Bird Age	Year 1	Year 2	Year 3	Year 4	Year 5
1-2	62	67	85	89	117
2-3	109	120	142	190	203
3-4	104	132	144	146	183

b. Write the Leslie matrix for this species of bird using the average values computed above (to **3 significant figures**). Use your Leslie matrix to estimate the population of each of the age classes for the next 5 years.

c. Find the eigenvalues and eigenvectors for this model, then give the limiting percent population in each of the age classes. What is the approximate annual rate of growth for this species of bird and how long would it take for the total population to double?

7. Consider the following predator-prey model

$$\begin{aligned} \dot{x} &= 0.2x - 0.02x^2 - 0.04xy \\ \dot{y} &= -0.3y + 0.09xy \end{aligned}$$

a. Which variable represents the predator and which one represents the prey in this model? Give a biological interpretation for each of the terms in the two equations above. What are the strengths and weaknesses for this model? (Give two of each.)

b. Use Maple to draw a phase portrait of this model. Show at least 3 representative trajectories in the phase portrait.

c. Find all equilibria for this model. Determine the eigenvalues and associated eigenvectors at each of the equilibria, then discuss the stability of these equilibria. Characterize each of the equilibria (*e.g.*, stable node, saddle node, unstable spiral).