Spring 2002 Math 337 Take-Home Exam 3

1. a. The Laplace transform becomes

$$\mathcal{L}\{\ddot{y}\} - 4\mathcal{L}\{\dot{y}\} = s^2\mathcal{L}\{y\} - sy(0) - \dot{y}(0) - 4\left(s\mathcal{L}\{y\} - y(0)\right) = \mathcal{L}\{20 - \cos(2t)\}$$

or

$$(s^2 - 4s)\mathcal{L}\{y\} = 2s - 9 + \frac{20}{s} - \frac{s}{s^2+4} = \frac{2s^4 - 9s^3 + 27s^2 - 36s + 80}{s(s^2+4)}.$$

The partial fractions decomposition needed is

$$\frac{2s^4 - 9s^3 + 27s^2 - 36s + 80}{s^2(s-4)(s^2+4)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-4} + \frac{Ds+E}{s^2+4},$$

where the coefficient $A = -5$ (by substituting $s = 0$), $C = \frac{19}{20}$ (by substituting $s = 4$), $D = \frac{1}{20}$ and $E = \frac{1}{5}$ (by substituting $s = 2i$ and simplifying the complex expression), and $B = 1$ (by finally examining s^4 terms). Thus,

$$\mathcal{L}\{y\} = \frac{-5}{s^2} + \frac{1}{s} + \frac{19}{20}\frac{1}{(s-4)} + \frac{1}{20}\frac{s}{(s^2+4)} + \frac{1}{10}\frac{2}{(s^2+4)},$$

or

$$y(t) = 1 - 5t + \frac{19}{20}e^{4t} + \frac{1}{20}\cos(2t) + \frac{1}{10}\sin(2t).$$

b. The Laplace transform becomes

$$\mathcal{L}\{\ddot{y}\} + 6\mathcal{L}\{\dot{y}\} + 9\mathcal{L}\{\dot{y}\} = s^2\mathcal{L}\{y\} - sy(0) - \dot{y}(0) + 6\left(s\mathcal{L}\{y\} - y(0)\right) + 9\mathcal{L}\{\dot{y}\} = \mathcal{L}\{18t^2 + 6e^{-3t}\}$$

or

$$(s^2 + 6s + 9)\mathcal{L}\{y\} = 3s + 16 + \frac{36}{s^3} + \frac{6}{s+3} = \frac{3s^5 + 25s^4 + 54s^3 + 36s + 108}{s^3(s+3)}.$$

The partial fractions decomposition needed is

$$\frac{3s^5 + 25s^4 + 54s^3 + 36s + 108}{s^3(s+3)^3} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{(s+3)^3} + \frac{E}{(s+3)^2} + \frac{F}{s+3},$$

where the coefficient $A = 4$ (by substituting $s = 0$), $D = 6$ (by substituting $s = -3$), $B = -\frac{8}{3}$ (since the coefficient of s^1 satisfies $36 = 27A + 27B$), $C = \frac{4}{3}$ (since the coefficient of s^2 satisfies $0 = 9A + 27B + 27C$), $F = \frac{5}{3}$ (since the coefficient of s^5 satisfies $3 = C + F$), and $E = \frac{17}{3}$ (since the coefficient of s^4 satisfies $25 = B + 9C + E + 6F$). Thus,

$$\mathcal{L}\{y\} = \frac{4}{s^3} - \frac{8}{3s^2} + \frac{4}{3s} + \frac{6}{(s+3)^3} + \frac{17}{3(s+3)^2} + \frac{5}{3(s+3)},$$

or

$$y(t) = 2t^2 - \frac{8}{3}t + \frac{4}{3} + 3t^2e^{-3t} + \frac{17}{3}te^{-3t} + \frac{5}{3}e^{-3t}.$$

c. The diffferential equation can be written

$$\ddot{y} + 4\dot{y} + 8y = 1 - u_1(t) - u_1(t-1) + u_2(t) + u_2(t)(t-2).$$

The Laplace transform becomes

$$\begin{aligned}\mathcal{L}\{\ddot{y}\} + 4\mathcal{L}\{\dot{y}\} + 8\mathcal{L}\{\dot{y}\} &= s^2\mathcal{L}\{y\} - sy(0) - \dot{y}(0) + 4\left(s\mathcal{L}\{y\} - y(0)\right) + 8\mathcal{L}\{\dot{y}\} \\ &= \mathcal{L}\{1 - u_1(t) - u_1(t-1) + u_2(t) + u_2(t)(t-2)\}\end{aligned}$$

or

$$(s^2+4s+8)\mathcal{L}\{y\} = -s-2+\frac{1}{s}-\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}+\frac{e^{-2s}}{s}+\frac{e^{-2s}}{s^2} = -s-2+\frac{1}{s}-\frac{e^{-s}(s+1)}{s^2}+\frac{e^{-2s}(s+1)}{s^2}.$$

The partial fractions decomposition needed are

$$\frac{1}{s(s^2+4s+8)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+8},$$

where the coefficient $A = \frac{1}{8}$ (by substituting $s = 0$), $B = -\frac{1}{8}$ (from the s^2 terms), and $C = -\frac{1}{2}$ (from the s^1 terms), and

$$\frac{s+1}{s^2(s^2+4s+8)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+4s+8},$$

where the coefficient $A = \frac{1}{8}$ (by substituting $s = 0$), $B = \frac{1}{16}$ (from the s^1 terms, $1 = 4A + 8B$), $C = -\frac{1}{16}$ (from the s^3 terms, $B + C = 0$), and $D = -\frac{3}{8}$ (from the s^2 terms, $A + 4B + D = 0$). Thus,

$$\mathcal{L}\{y\} = -\frac{s+2}{(s+2)^2+4}+\frac{1}{8s}-\frac{1}{8}\frac{s+2+2}{(s+2)^2+4}-e^{-s}\left(\frac{\frac{1}{8s^2}+\frac{1}{16s}-\frac{1}{16}s+2+4}{(s+2)^2+4}\right)+e^{-2s}\left(\frac{\frac{1}{8s^2}+\frac{1}{16s}-\frac{1}{16}s+2+4}{(s+2)^2+4}\right).$$

or

$$\begin{aligned}y(t) &= \frac{1}{8}-\frac{9}{8}e^{-2t}\cos(2t)-\frac{1}{8}e^{-2t}\sin(2t) \\ &\quad -u_1(t)\left(\frac{1}{8}(t-1)+\frac{1}{16}-\frac{1}{16}e^{-2(t-1)}\cos(2(t-1))-\frac{1}{8}e^{-2(t-1)}\sin(2(t-1))\right) \\ &\quad +u_2(t)\left(\frac{1}{8}(t-2)+\frac{1}{16}-\frac{1}{16}e^{-2(t-2)}\cos(2(t-2))-\frac{1}{8}e^{-2(t-2)}\sin(2(t-2))\right).\end{aligned}$$

2. a. The diffferential equation can be written

$$\ddot{y} + y = \delta(t) + \delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi) + ...$$

The Laplace transform becomes

$$\begin{aligned}\mathcal{L}\{\ddot{y}\} + \mathcal{L}\{\dot{y}\} &= s^2\mathcal{L}\{y\} - sy(0) - \dot{y}(0) + \mathcal{L}\{\dot{y}\} \\ &= \mathcal{L}\{\delta(t) + \delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi) + ...\}\end{aligned}$$

or

$$(s^2+1)\mathcal{L}\{y\} = 1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + ...$$

Thus,

$$\mathcal{L}\{y\} = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} + \frac{e^{-3\pi s}}{s^2+1} + ...$$

or

$$\begin{aligned} y(t) &= \sin(t) + u_\pi(t)\sin(t-\pi) + u_{2\pi}(t)\sin(t-2\pi) + u_{3\pi}(t)\sin(t-3\pi) + ... \\ &= \sin(t)(1 - u_\pi(t) + u_{2\pi}(t) - u_{3\pi}(t) + ...) = \sin(t)\sum_{n=0}^{\infty}(-1)^n u_{n\pi}(t). \end{aligned}$$

b. The diffferential equation can be written

$$\ddot{y} + y = \delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + ...$$

The Laplace transform becomes

$$\begin{aligned} \mathcal{L}\{\ddot{y}\} + \mathcal{L}\{\dot{y}\} &= s^2\mathcal{L}\{y\} - sy(0) - \dot{y}(0) + \mathcal{L}\{\dot{y}\} \\ &= \mathcal{L}\{\delta(t) + \delta(t-2\pi) + \delta(t-4\pi) + \delta(t-6\pi) + ...\} \end{aligned}$$

or

$$(s^2+1)\mathcal{L}\{y\} = 1 + e^{-2\pi s} + e^{-4\pi s} + e^{-6\pi s} + ...$$

Thus,

$$\mathcal{L}\{y\} = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} + \frac{e^{-4\pi s}}{s^2+1} + \frac{e^{-6\pi s}}{s^2+1} + ...$$

or

$$\begin{aligned} y(t) &= \sin(t) + u_{2\pi}(t)\sin(t-2\pi) + u_{4\pi}(t)\sin(t-4\pi) + u_{6\pi}(t)\sin(t-6\pi) + ... \\ &= \sin(t)(1 + u_{2\pi}(t) + u_{4\pi}(t) + u_{6\pi}(t) + ...) = \sin(t)\sum_{n=0}^{\infty} u_{2n\pi}(t). \end{aligned}$$

c. In the first case, sine functions alternately cancel each other out. Thus, on alternate intervals of π, the function is positive part of the sine curve followed by an interval of the zero function. The second case is the resonance case. In this case, each interval of 2π has the solution increasing the amplitude of the sine function by 1. This solution becomes unbounded, so the bridge would collapse.

3. a. If $\mathcal{L}\{f\} = F(s)$, then

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Leibnitz's rule of differentiation of the integral states that

$$\frac{d}{dx}\int_a^b f(x,y)\,dy = \int_a^b \frac{\partial f(x,y)}{\partial x}dy.$$

In other words, the differentiation operator can commute with the integral. Thus,

$$\frac{dF(s)}{ds} = \int_0^\infty \frac{\partial f(t)e^{-st}}{\partial s}dt = -\int_0^\infty t\,f(t)e^{-st}dt.$$

It follows that

$$\mathcal{L}\{t\,f(t)\} = -\frac{dF(s)}{ds}.$$

b. Since $\mathcal{L}\sin(\omega t) = \frac{\omega}{s^2+\omega^2}$, then

$$-\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2s\omega}{(s^2+\omega^2)^2},$$

which is $\mathcal{L}\{t\,\sin(\omega t)\}$.

4. a. Let f be a periodic function with period T (with $T > 0$), then $f(t+T) = f(t)$ for all $t \geq 0$. We can write

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty f(t)e^{-st}dt \\
&= \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \int_{2T}^{3T} f(t)e^{-st}dt + ...
\end{aligned}$$

By a change of variables $u = t - nT$, we can write

$$\int_{nT}^{(n+1)T} f(t)e^{-st}dt = \int_0^T f(u+nT)e^{-s(u+nT)}du = e^{-snT}\int_0^T f(u)e^{-su}du,$$

using the periodicity of $f(t)$. It follows that the infinite sum above can be written

$$\mathcal{L}\{f(t)\} = \left(1 + e^{-sT} + e^{-2sT} + e^{-3sT} + ...\right)\int_0^T f(t)e^{-st}dt.$$

The expression in front of the integral is a geometric series which adds to $1/(1-e^{-sT})$, so

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt.$$

b. From the result above, the Laplace transform of the saw tooth function is given by

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-s}}\int_0^1 t\,e^{-st}dt \\
&= \frac{1}{1-e^{-s}}\left[-\frac{te^{-st}}{s}\bigg|_0^1 + \frac{1}{s}\int_0^1 e^{-st}dt\right] \\
&= \frac{1}{1-e^{-s}}\left(-\frac{e^{-s}}{s} - \frac{e^{-st}}{s^2}\bigg|_0^1\right) \\
&= \frac{1}{1-e^{-s}}\left(\frac{1}{s^2} - \frac{(s+1)e^{-s}}{s^2}\right)
\end{aligned}$$