

1. a. The eigenvalues and eigenvectors for the matrix are

$$\lambda_1 = 1, \quad X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -3, \quad X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

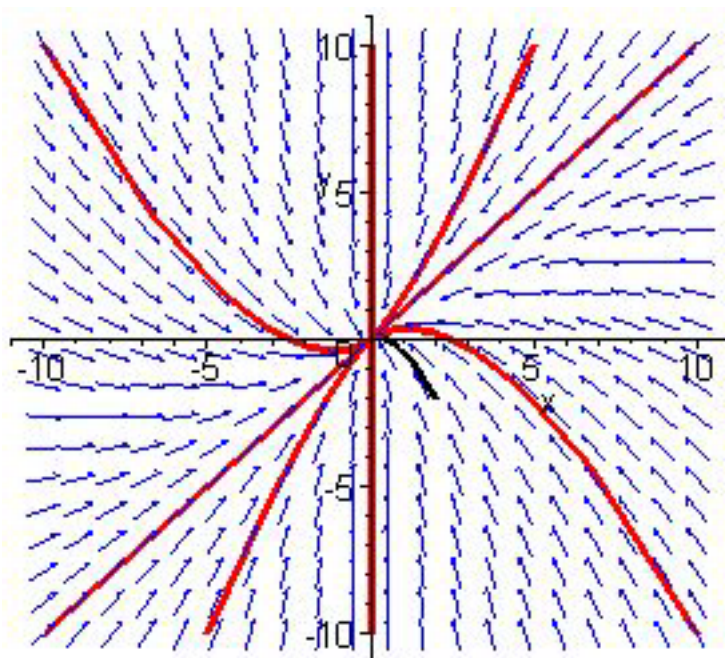
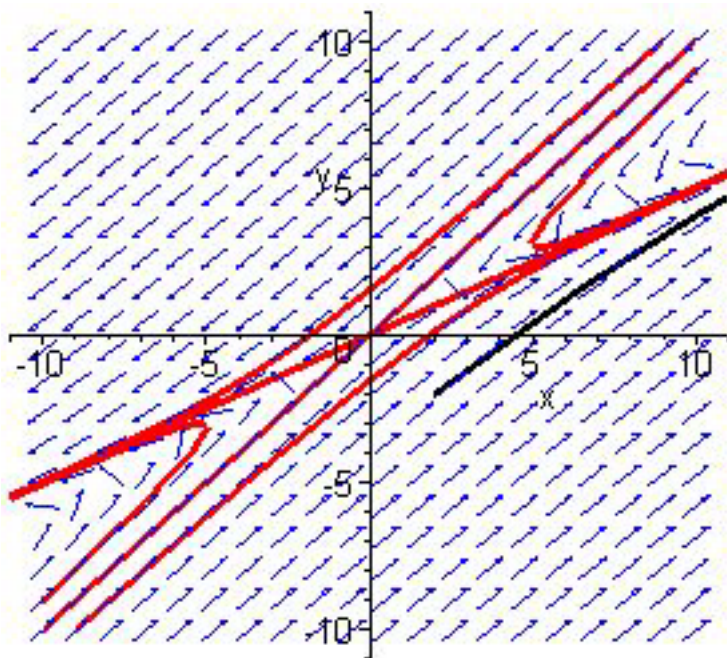
The general solution is given by

$$Y(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

The solution to the initial value problem is given by

$$Y(t) = 4 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t - 6 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

The equilibrium, $(0, 0)$, is a saddle point. Below to the left is the phase portrait.



b. The eigenvalues and eigenvectors for the matrix are

$$\lambda_1 = -2, \quad X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_2 = -3, \quad X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The general solution is given by

$$Y(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}.$$

The solution to the initial value problem is given by

$$Y(t) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} - 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}.$$

The equilibrium, $(0, 0)$, is stable node. Above to the right is the phase portrait.

c. The only eigenvalue and eigenvector for the matrix is

$$\lambda = 0, \quad X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

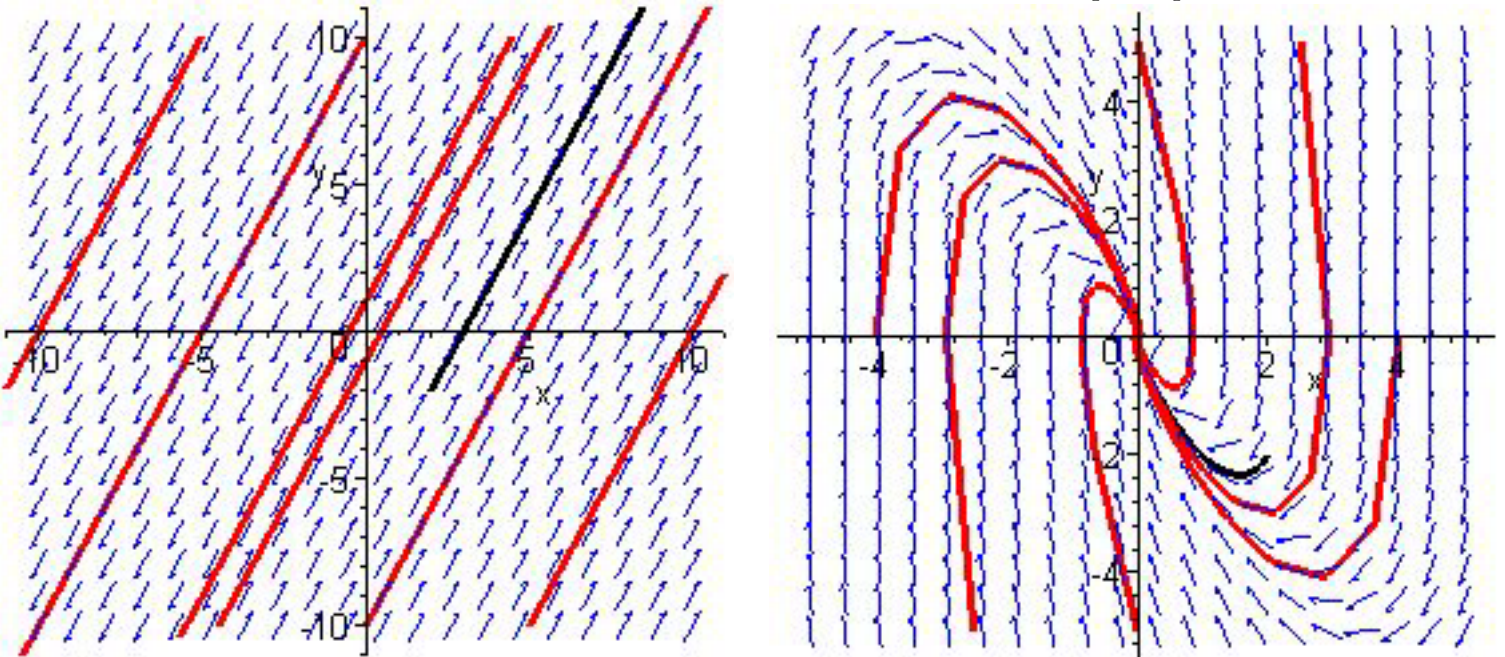
The general solution is given by

$$Y(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 2x_0 - y_0 \\ 4x_0 - 2y_0 \end{pmatrix} t.$$

The solution to the initial value problem is given by

$$Y(t) = \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 6 \\ 12 \end{pmatrix} t.$$

The equilibria are all points satisfying $y_2 = 2y_1$. These are all unstable equilibria with all solutions away from the equilibria moving parallel to this line. Below to the left is the phase portrait.



2. a. The solution to the initial value problem is

$$y(t) = e^{-2t} \left(2 \cos(\sqrt{2}t) + \sqrt{2} \sin(\sqrt{2}t) \right).$$

c. The direction field and the solution (in black) to the initial value problem are shown above to the right.

b. The first order system is given by

$$\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -6 & -4 \end{pmatrix} Y.$$

The eigenvalues and eigenvectors for the matrix are

$$\lambda_1 = -2 + i\sqrt{2}, \quad X_1 = \begin{pmatrix} 1 \\ -2 + i\sqrt{2} \end{pmatrix} \quad \text{and} \quad \lambda_2 = -2 - i\sqrt{2}, \quad X_2 = \begin{pmatrix} 1 \\ -2 - i\sqrt{2} \end{pmatrix}.$$

The general solution is given by

$$Y(t) = e^{-2t} \left(c_1 \begin{pmatrix} \cos(\sqrt{2}t) \\ -2 \cos(\sqrt{2}t) - \sqrt{2} \sin(\sqrt{2}t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\sqrt{2}t) \\ -2 \sin(\sqrt{2}t) + \sqrt{2} \cos(\sqrt{2}t) \end{pmatrix} \right).$$

The equilibrium, $(0, 0)$, is a stable spiral.

3. a. The solution to the differential equation

$$\frac{dy}{dt} = -2y, \quad y(1) = 4$$

is

$$y(t) = 4e^{-2(t-1)}.$$

The linear differential equation in $x(t)$ becomes

$$\frac{dx}{dt} + \frac{x}{t} = 4e^{-2(t-1)}, \quad x(1) = -3,$$

which has an integrating factor

$$\mu(t) = \exp\left(\int \frac{dt}{t}\right) = t.$$

Thus,

$$\frac{d}{dt}(tx(t)) = 4te^{-2(t-1)}.$$

It follows that

$$\begin{aligned} tx(t) &= 4 \int te^{-2(t-1)} dt = -(2t+1)e^{-2(t-1)} + C \\ x(t) &= -\frac{1}{t} \left((2t+1)e^{-2(t-1)} - C \right) \end{aligned}$$

The initial condition, $x(1) = -3$ implies that $C = 0$, so the solution to the system of equations is

$$\begin{aligned} x(t) &= -\frac{(2t+1)e^{-2(t-1)}}{t} \\ y(t) &= 4e^{-2(t-1)}. \end{aligned}$$

b. The Euler's formula for this 2-D system is given by

$$\begin{aligned} x_{n+1} &= x_n + h \left(-\frac{x_n}{t_n} + y_n \right), \\ y_{n+1} &= y_n - 2hy_n. \end{aligned}$$

Below is a table showing the values of the Euler iterates when $h = 0.1$.

t_n	x_n	y_n
$t_0 = 1$	$x_0 = -3$	$y_0 = 4$
$t_1 = 1.1$	$x_1 = -2.3$	$y_1 = 3.2$
$t_2 = 1.2$	$x_2 = -1.770909091$	$y_2 = 2.56$
$t_3 = 1.3$	$x_3 = -1.367333333$	$y_3 = 2.048$
$t_4 = 1.4$	$x_4 = -1.057353846$	$y_4 = 1.6384$
$t_5 = 1.5$	$x_5 = -0.817988571$	$y_5 = 1.31072$
$t_6 = 1.6$	$x_6 = -0.632384$	$y_6 = 1.048576$
$t_7 = 1.7$	$x_7 = -0.4880024$	$y_7 = 0.8388608$
$t_8 = 1.8$	$x_8 = -0.375410296$	$y_8 = 0.67108864$
$t_9 = 1.9$	$x_9 = -0.287445305$	$y_9 = 0.536870912$
$t_{10} = 2$	$x_{10} = -0.218629513$	$y_{10} = 0.42949673$

Below is a table showing the values of the Euler iterates when $h = 0.05$.

t_n	x_n	y_n
$t_0 = 1$	$x_0 = -3$	$y_0 = 4$
$t_1 = 1.05$	$x_1 = -2.65$	$y_1 = 3.6$
$t_2 = 1.1$	$x_2 = -2.343809524$	$y_2 = 3.24$
$t_3 = 1.15$	$x_3 = -2.075272727$	$y_3 = 2.916$
$t_4 = 1.2$	$x_4 = -1.839243478$	$y_4 = 2.6244$
$t_5 = 1.25$	$x_5 = -1.631388333$	$y_5 = 2.36196$
$t_6 = 1.3$	$x_6 = -1.4480348$	$y_6 = 2.125764$
$t_7 = 1.35$	$x_7 = -1.286052954$	$y_7 = 1.9131876$
$t_8 = 1.4$	$x_8 = -1.142761983$	$y_8 = 1.72186884$
$t_9 = 1.45$	$x_9 = -1.015855613$	$y_9 = 1.549681956$
$t_{10} = 1.5$	$x_{10} = -0.903342011$	$y_{10} = 1.39471376$
$t_{11} = 1.55$	$x_{11} = -0.803494923$	$y_{11} = 1.255242384$
$t_{12} = 1.6$	$x_{12} = -0.714813613$	$y_{12} = 1.129718146$
$t_{13} = 1.65$	$x_{13} = -0.63598978$	$y_{13} = 1.016746331$
$t_{14} = 1.7$	$x_{14} = -0.565880046$	$y_{14} = 0.915071698$
$t_{15} = 1.75$	$x_{15} = -0.50348293$	$y_{15} = 0.823564528$
$t_{16} = 1.8$	$x_{16} = -0.447919477$	$y_{16} = 0.741208076$
$t_{17} = 1.85$	$x_{17} = -0.398416866$	$y_{17} = 0.667087268$
$t_{18} = 1.9$	$x_{18} = -0.354294479$	$y_{18} = 0.600378541$
$t_{19} = 1.95$	$x_{19} = -0.314952013$	$y_{19} = 0.540340687$
$t_{20} = 2$	$x_{20} = -0.279859286$	$y_{20} = 0.486306618$

c. The exact solution at $t = 2$ is $x(2) = -\frac{5}{2}e^{-2} = -0.3383382$ and $y(2) = 4e^{-2} = 0.5413411$. The difference between the Euler approximate solutions when $h = 0.1$ is about 0.12 for x and 0.11 for y . When the stepsize is decreased to $h = 0.05$, then the difference between the Euler approximate solutions is about 0.06 for x and 0.055 for y or about half the error. In general, for Euler's method, if you halve the stepsize, then you halve the error.

4. a. Model A is given by

$$\begin{aligned}\frac{dx}{dt} &= 0.2x - 0.004x^2 - 0.008xy, \\ \frac{dy}{dt} &= 0.1y - 0.006xy - 0.002y^2,\end{aligned}$$

and has the four equilibria given by

$$(x_e, y_e) = (0, 0), (50, 0), (0, 50), (10, 20)$$

with the first being the trivial equilibrium, the second being the carrying capacity for Species x , the third being the carrying capacity for Species y , and the last being the co-existence equilibrium. Model B is given by

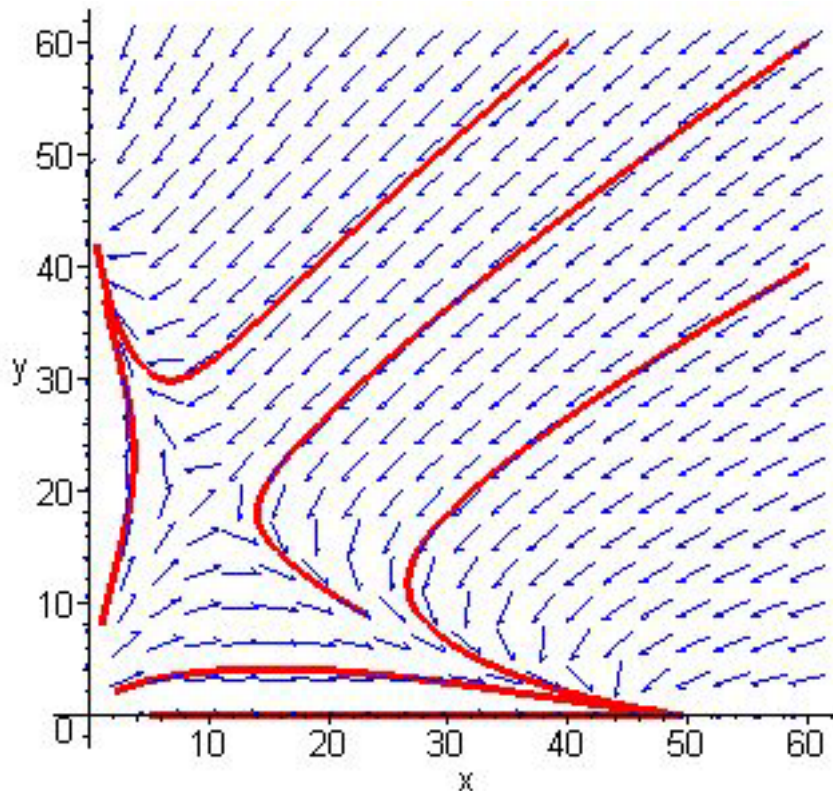
$$\begin{aligned}\frac{dx}{dt} &= 0.1x - 0.01x^2 - 0.005xy, \\ \frac{dy}{dt} &= 0.2y - 0.002xy - 0.004y^2.\end{aligned}$$

and has the four equilibria given by

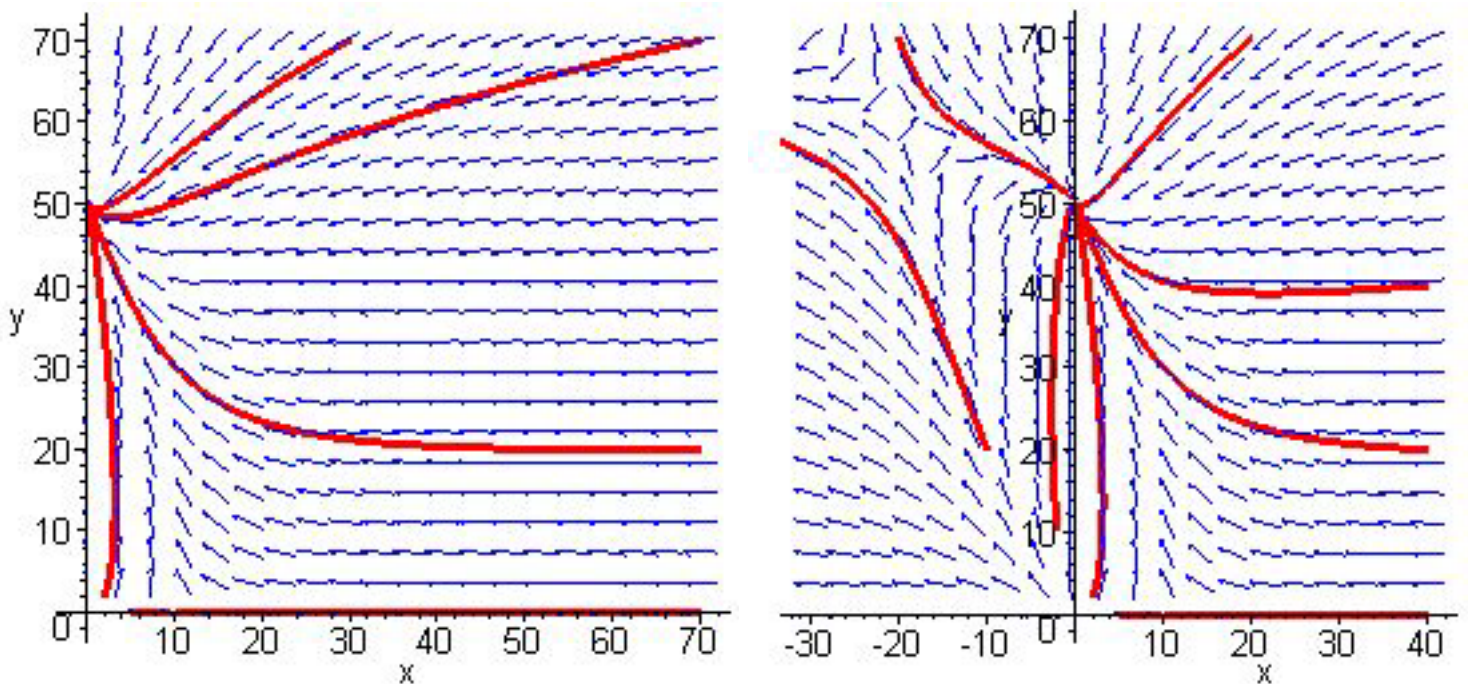
$$(x_e, y_e) = (0, 0), (10, 0), (0, 50), (-20, 60)$$

with the first three being the trivial and carrying capacity equilibria similar to Model A and the last equilibrium not biologically possible because of the negative population.

b. Below is the direction field with some representative solutions for Model A.



Below are two direction fields with some representative solutions for Model B. The one on the left shows only the non-negative domain (appropriate for populations), while the one on the right shows the domain including all equilibria.



c. Both Models have the equilibrium $(0,0)$ as an unstable node (source). For Model A, the single species equilibria $(50,0)$ and $(0,50)$ are stable nodes (sinks), while the co-existence equilibrium $(10,20)$ is a saddle node. For Model B, the single species equilibrium $(10,0)$ is a saddle node, while the other single species equilibrium $(0,50)$ is a stable node (sink). The equilibrium outside the domain $(-20,60)$ is another saddle node.

d. In both models, one species eventually dominates driving the other one to extinction. In Model A, Species x has a slight advantage, so that when the populations begin with equal initial populations (either large or small), then eventually, the population tends to the equilibrium $(50,0)$. In Model B, Species y always dominates and drives Species x to extinction, except when only Species x exists and the population goes to the equilibrium $(10,0)$. When only a small population of one of the species exists, then both models have the populations going to the carrying capacity equilibrium for that species.

5. Suppose that $\lambda = 0$ is an eigenvalue of \mathbf{A} . By the properties of eigenvalues,

$$\det |\mathbf{A} - \lambda \mathbf{I}| = \det |\mathbf{A} - 0 \cdot \mathbf{I}| = \det |\mathbf{A}| = 0.$$

Thus, $\det |\mathbf{A}| = 0$.

b. Suppose that $\det |\mathbf{A}| = 0$. It follows that

$$0 = \det |\mathbf{A}| = \det |\mathbf{A} - 0 \cdot \mathbf{I}|.$$

Thus, $\lambda = 0$ is an eigenvalue of \mathbf{A} .