Spring 2002 Math 337 Take-Home Exam 1 - Solutions

1. a. The differential equation

$$\frac{dy}{dt} = 5 - 0.2y \qquad \text{or} \qquad \frac{dy}{dt} = -0.2(y - 25)$$

uses the substitution $z(t) = y(t) - 25$. Thus, we solve the differential equation

$$\frac{dz}{dt} = -0.2z, \quad z(2) = y(2) - 25 = 10 - 25 = -15.$$

Thus,

$$z(t) = ce^{-0.2t} \qquad \text{with} \qquad -15 = ce^{-0.4} \qquad \text{or} \qquad c = -15e^{0.4}$$

So,

$$y(t) - 25 = -15e^{-0.2(t-2)} \qquad \text{or} \qquad y(t) = 25 - 15e^{-0.2(t-2)}.$$

b. The differential equation

$$\frac{dy}{dt} = e^{-t}\left(1 + y^2\right)$$

is solved by separation of variables as follows:

$$\int \frac{dy}{1+y^2} = \int e^{-t}dt \qquad \text{or} \qquad \arctan(y(t)) = -e^{-t} + C.$$

From the initial condition, $y(0) = 0$,

$$C = \arctan(0) + e^0 = 1.$$

It follows that the solution is given by

$$y(t) = \tan\left(1 - e^{-t}\right).$$

c. The differential equation

$$\frac{dy}{dt} = t^3 - 2ty \qquad \text{which can be written} \qquad \frac{dy}{dt} + 2ty = t^3$$

is a linear differential equation. Its integrating factor is

$$\mu(t) = exp\left(\int 2tdt\right) = e^{t^2}.$$

Thus,

$$\frac{d}{dt}\left(e^{t^2}y(t)\right) = t^3e^{t^2}$$

or

$$e^{t^2}y(t) = \int t^3e^{t^2}dt + C = \frac{e^{t^2}}{2}\left(t^2 - 1\right) + C.$$

It follows that

$$y(t) = \frac{1}{2}\left(t^2 - 1\right) + Ce^{-t^2}.$$

The initial condition $y(0) = 10$ gives $10 = -\frac{1}{2} + C$ or $C = \frac{21}{2}$,

$$y(t) = \frac{1}{2}\left(t^2 - 1\right) + \frac{21}{2}e^{-t^2}.$$

d. The differential equation

$$\frac{dy}{dt} = \frac{y^2 - 2ty}{t^2} \qquad \text{which can be written} \qquad \frac{dy}{dt} + \frac{2}{t}y = \frac{y^2}{t^2}$$

is a Bernoulli equation, which can be solved using the substitution $z(t) = y^{-1}(t)$. (However, I was expecting to have you simply solve this with Maple.) Note that $z'(t) = -y^{-2}y'(t)$, so after multiplying the above equation by $-y^{-2}$, we have

$$\frac{dz}{dt} - \frac{2}{t}z = -\frac{1}{t^2}.$$

This is a linear differential equation in $z(t)$ with an integrating factor of

$$\mu(t) = exp\left(-2\int \frac{dt}{t}\right) = \frac{1}{t^2}.$$

Thus,

$$\frac{d}{dt}\left(\frac{z(t)}{t^2}\right) = -\frac{1}{t^4}$$

or

$$\frac{z(t)}{t^2} = -\int \frac{dt}{t^4} + C = \frac{1}{3t^3} + C.$$

It follows that

$$z(t) = \frac{1}{3t} + Ct^2 = \frac{1}{y(t)}.$$

It follows that

$$y(t) = \frac{3t}{1 + 3Ct^3}.$$

The initial condition $y(1) = 2$ gives $2 = \frac{3}{1+3C}$ or $3C = \frac{1}{2}$, so

$$y(t) = \frac{3t}{1 + \frac{1}{2}t^3} = \frac{6t}{2 + t^3}.$$

2. a. The differential equation

$$\frac{dy}{dt} = y - t^2 \qquad \text{which can be written} \qquad \frac{dy}{dt} - y = -t^2$$

is a linear differential equation. Its integrating factor is

$$\mu(t) = exp\left(-\int dt\right) = e^{-t}.$$

Thus,

$$\frac{d}{dt}\left(e^{-t}y(t)\right) = -t^2e^{-t}$$

or

$$e^{-t}y(t) = -\int t^2e^{-t}dt + C = e^{-t}\left(t^2+2t+2\right) + C.$$

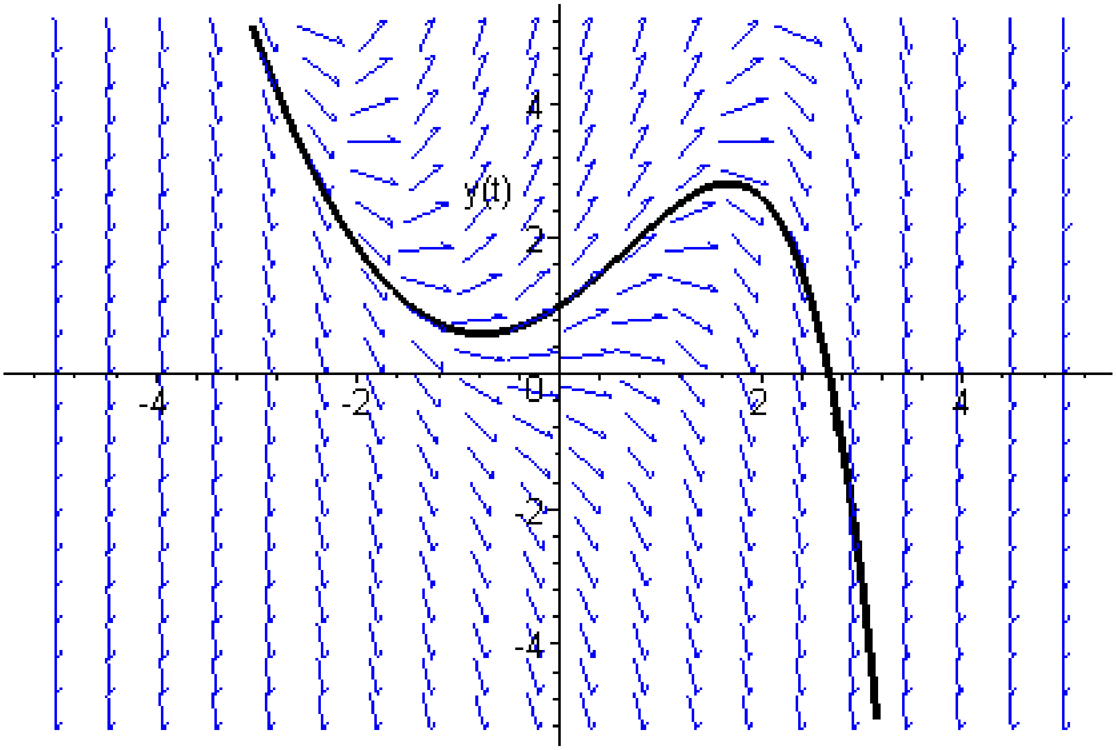
It follows that

$$y(t) = t^2 + 2t + 2 + Ce^t.$$

The initial condition $y(0) = 1$ gives $1 = 2 + C$ or $C = -1$,

$$y(t) = t^2 + 2t + 2 - e^t.$$

b. The slope field for $t \in [-5, 5]$ and $y \in [-5, 5]$ with the solution above is shown below.



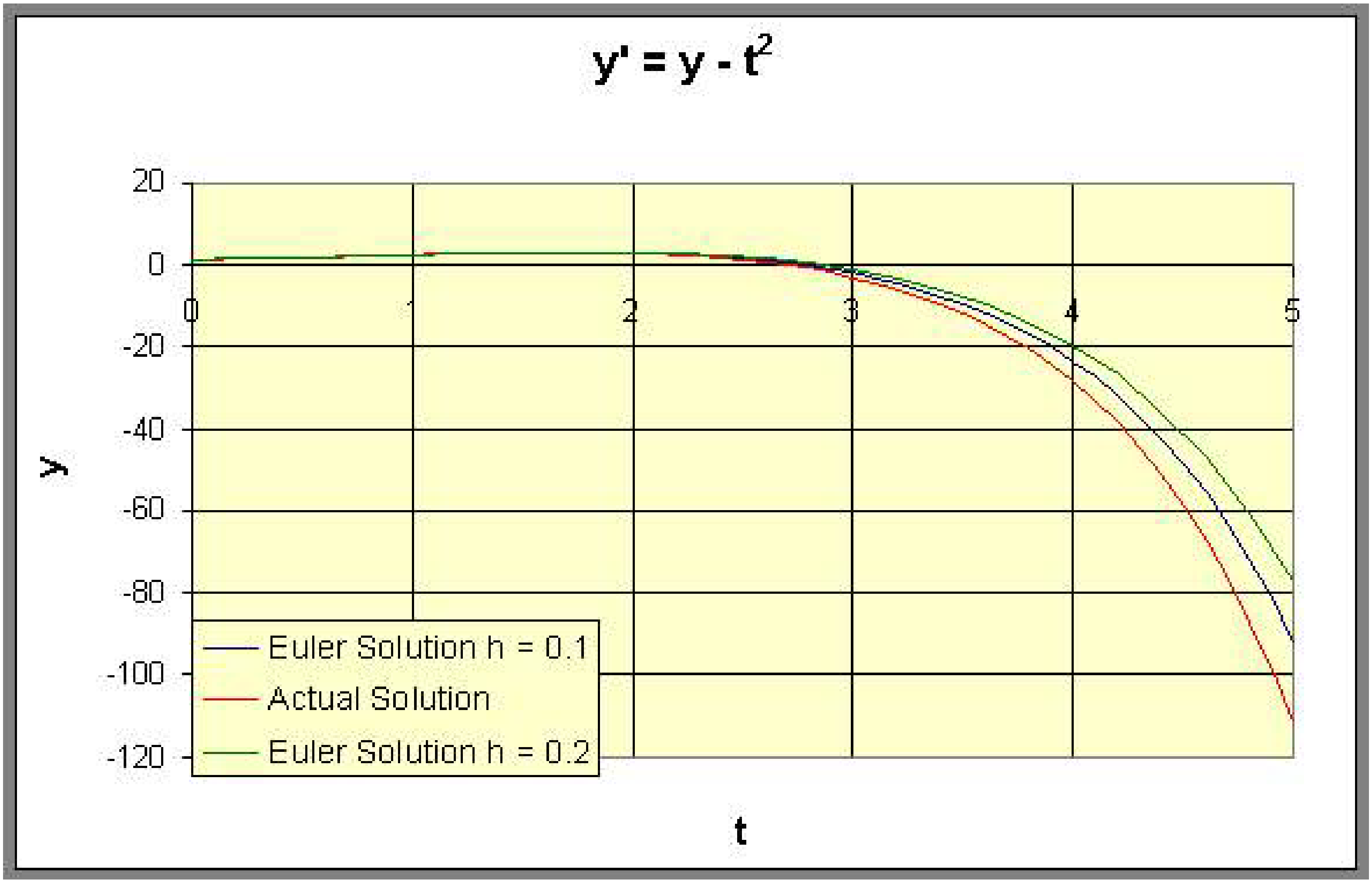
c. Euler's formula for the differential equation is

$$y_{n+1} = y_n + h(y_n - t_n^2),$$

starting with the initial condition $y_0 = 1$ and solving for $t \in [0, 5]$. The calculations were done in Excel, and the table below summarizes the values for stepsizes of $h = 0.2$ and $h = 0.1$ at the times $t = 1, 2, 3, 4$, and 5.

t	$h = 0.2$	$h = 0.1$	actual
$t = 1$	$y = 2.214016$	$y = 2.246883294$	$y = 2.281718172$
$t = 2$	$y = 2.769916293$	$y = 2.699750056$	$y = 2.610943901$
$t = 3$	$y = -1.28842589$	$y = -2.094342496$	$y = -3.085536923$
$t = 4$	$y = -19.80511991$	$y = -23.68518112$	$y = -28.59815003$
$t = 5$	$y = -77.27545997$	$y = -92.02993817$	$y = -111.4131591$

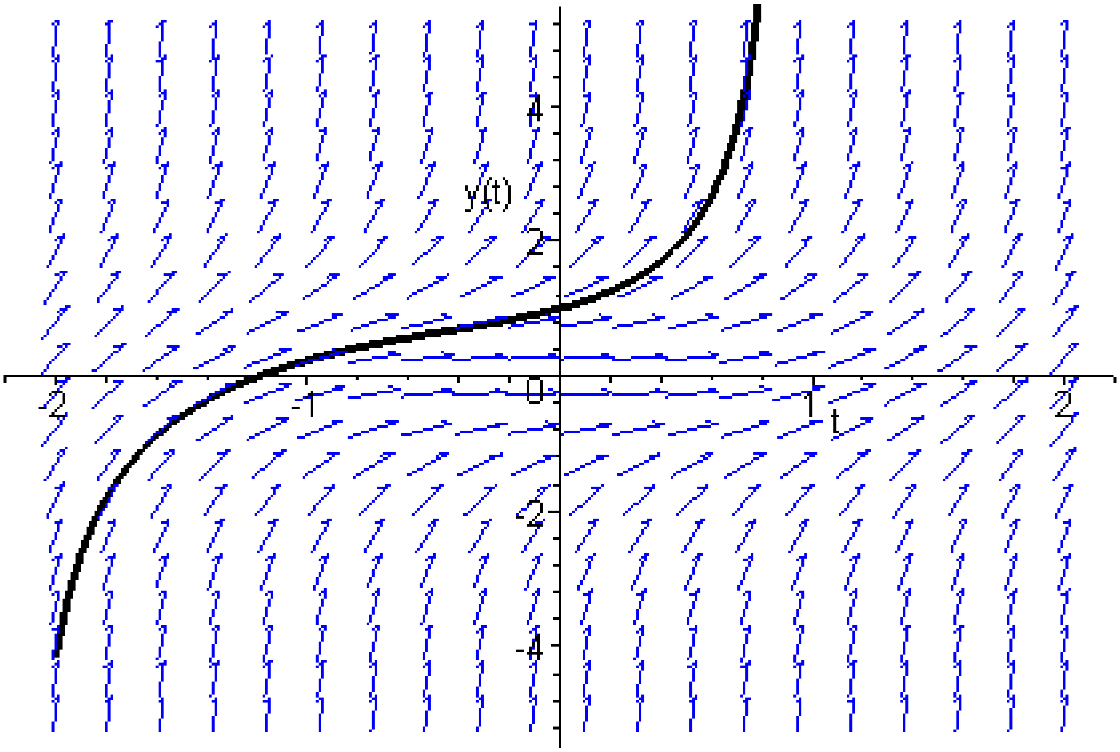
The graph of the Euler and actual solutions is below.



3. a. The slope field for the differential equation

$$\frac{dy}{dt} = y^2 + t^2$$

with the specific solution satisfying $y(0) = 1$ is shown below for $t \in [-2, 2]$ and $y \in [-5, 5]$.



b. Maple finds the solution to the initial value problem, and it involves many Bessel functions as is given below:

$$y(t) = -\frac{t\left(-\frac{\left((\Gamma(3/4))^2-\pi\right)BesselJ(-3/4,1/2\,t^2)}{(\Gamma(3/4))^2} + BesselY\left(-3/4, 1/2\,t^2\right)\right)}{\left(-\frac{\left((\Gamma(3/4))^2-\pi\right)BesselJ(1/4,1/2\,t^2)}{(\Gamma(3/4))^2} + BesselY\left(1/4, 1/2\,t^2\right)\right)^{-1}}.$$

By using Maple's fsolve command with this solution (solving when $1/y(t) = 0$), we find that the solution becomes unbounded at $t = 0.9698106539$.

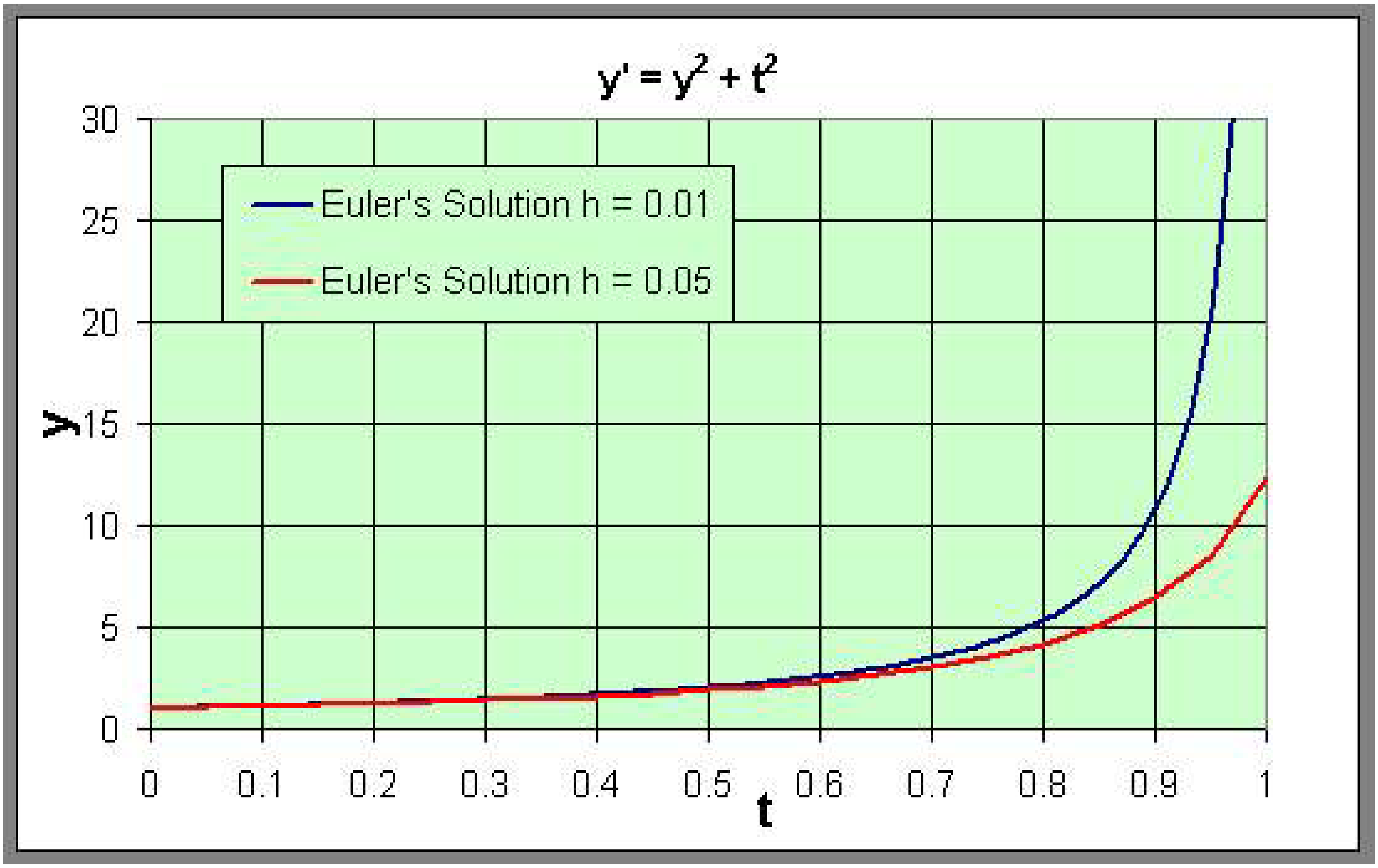
c. Euler's formula for the differential equation is

$$y_{n+1} = y_n + h(y_n^2 + t_n^2),$$

starting with the initial condition $y_0 = 1$ and solving for $t \in [0, 1]$. The calculations were done in Excel, and the table below summarizes the values for stepsizes of $h = 0.05$ and $h = 0.01$ at the times $t = 0.5, 0.8$, and 0.9.

t	$h = 0.05$	$h = 0.01$	actual
$t = 0.5$	$y = 1.933180859$	$y = 2.035994166$	$y = 2.0669997$
$t = 0.8$	$y = 4.201314965$	$y = 5.342755727$	$y = 5.8486168$
$t = 0.9$	$y = 6.460597267$	$y = 10.80408753$	$y = 14.304864$

The graph of the Euler solutions is below.



4. a. For the differential equation $\frac{dy}{dt} = \alpha y - y^2 = y(\alpha - y)$ with the parameter α, there are equilibria at $y = 0$ and $y = \alpha$. This produces a transcritical bifurcation at $\alpha = 0$. When $\alpha = 0$, there is only the one equilibrium $y = 0$, which is a node. For $\alpha < 0$, the equilibrium $y = \alpha$ is a source, while the equilibrium $y = 0$ is a sink. For $\alpha > 0$, the equilibrium $y = 0$ is a source, while the equilibrium $y = \alpha$ is a sink.

b. For the differential equation $\frac{dy}{dt} = -y + \tanh(\alpha y)$ with the parameter α, there are equilibria at $y = 0$ and solutions of $y = \tanh(\alpha y)$. The latter has nonzero solutions (2 by symmetry) if and only if $\alpha > 1$. Thus, there is a pitchfork bifurcation at $\alpha = 1$. When $\alpha \leq 1$, there is only the one equilibrium $y = 0$, which is a sink. For $\alpha > 1$, the equilibrium $y = 0$ is a source, while the other two equilibria are sinks.

5. a. We begin by finding the change in amount of pesticide entering and leaving the lake (conservation of mass). Let $A(t)$ be the amount of pesticide in the lake, then the concentration is given by $c(t) = A(t)/V$, where the volume remains constant at $V = 10^5$ m^3. The rate of change in amount

is given by

$$\frac{dA}{dt} = f_1Q_1f + f_2Q_2 - f_3c(t) = 1400 + 600 - 500c(t),$$

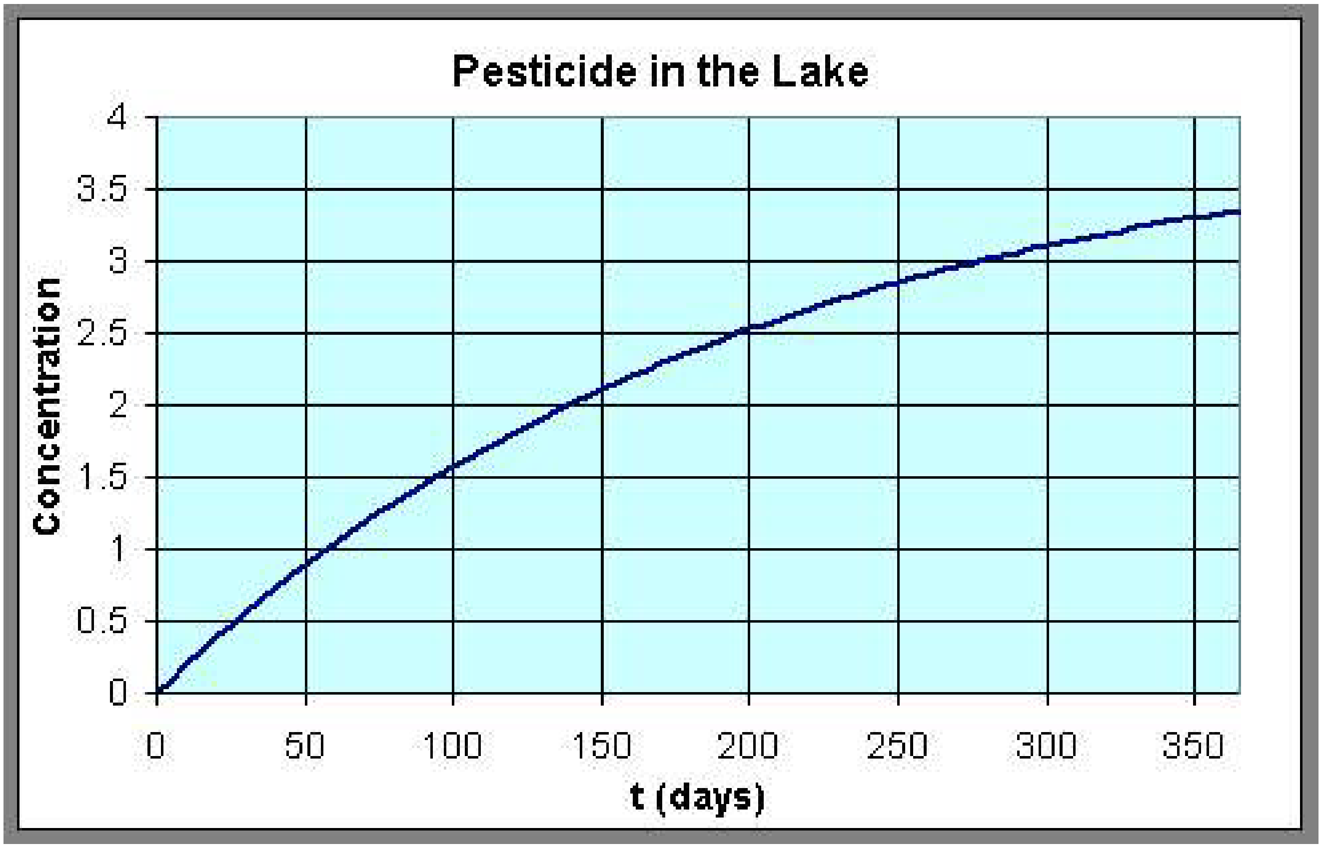
where the units are μg/day. By dividing the above equation by the volume, the differential equation for the concentration is given by

$$\frac{dc}{dt} = 0.02 - 0.005c = -0.005(c-4), \qquad c(0) = 0.$$

This equation is solved by the substitution $z(t) = c(t) - 4$, so $z' = -0.005z$ with $z(0) = -4$. The solution is

$$z(t) = -4e^{-0.005t} = c(t) - 4 \qquad \text{or} \qquad c(t) = 4 - 4e^{-0.005t}.$$

b. We solve $2 = 4 - 4e^{-0.005t}$, giving $e^{0.005t} = 2$ or $t = 200\ln(2) \simeq 138.6$ days. The limiting concentration of the pesticide is $4\,\mu\text{g/m}^3$. A graph for a year is shown below.



6. a. The Malthusian growth model is given by $P(t) = 76e^{rt}$, where t is in years after 1900 and P is in millions. Given that the population was 105.7 million in 1920, we have

$$105.7 = 76e^{20r} \qquad \text{or} \qquad r = 0.05\ln\left(\frac{105.7}{76}\right) \simeq 0.01649.$$

A population growing at this rate would double in 42.0 years.

b. From the model above, $P(10) = 89.6$ and $P(50) = 173.4$, so the percent error in 1910 is $100(92.0 - 89.6)/92.0 = 2.6\%$, while the percent error in 1950 is $100|151.3 - 173.4|/151.3 = 14.6\%$.

c. A logistic growth model for the U. S. population is a Bernoulli's equation, so a substitution is the easiest method to solve the equation. Let $z(t) = P^{-1}(t)$, so $\frac{dz}{dt} = -P^{-2}(t)\frac{dP}{dt}$. If we multiply the logistic growth equation by $-P^{-2}(t)$, then

$$-P^{-2}(t)\frac{dP}{dt} = -0.02P^{-1}\left(1 - \frac{P}{420}\right) \qquad \text{or} \qquad \frac{dz}{dt} = -0.02\left(z - \frac{1}{420}\right)$$

The equation is $z(t)$ is solved by another substitution, $w(t) = z(t) - \frac{1}{420}$. Since $P(0) = 76.0$, it follows that $z(0) = \frac{1}{76.0}$ and $w(0) = \frac{1}{76.0} - \frac{1}{420} = 0.010777$. Hence,

$$w(t) = 0.010777e^{-0.02t} = z(t) - \frac{1}{420} \qquad \text{or} \qquad z(t) = 0.010777e^{-0.02t} + \frac{1}{420}.$$

Thus,

$$P(t) = \frac{420}{1 + 4.5263e^{-0.02t}}.$$

From this model, $P(10) = 89.3$, $P(20) = 104.1$ and $P(50) = 157.6$, so the percent error in 1910 is $100|92.0 - 89.3|/92.0 = 2.9\%$, the percent error in 1920 is $100|105.7 - 104.1|/105.7 = 1.5\%$, and the percent error in 1950 is $100|151.3 - 157.6|/151.3 = 4.2\%$. From the 1900 population, the logistic growth model predicts 47.1 years for the population to double (5 more than the Malthusian growth model).

d. The equilibria for this logistic growth model are $P = 0$ (a source) and $P = 420$ million (a sink). The second of these equilibria is attracting, so the U. S. population would be predicted to level off at $P = 420$ million.