Find the derivatives of the following functions:

1. $f(x) = \dfrac{x^3 - \ln(x)}{1 - x^2} + \dfrac{2}{x^2}$,

2. $f(x) = \dfrac{x^2 - e^{-x}}{3x + 1} + xe^{-x}$,

3. $f(x) = \dfrac{\sqrt{x}}{2 + x} - \dfrac{1}{e^{3x}}$,

4. $f(x) = \dfrac{x^2 + 5}{x^2 - e^x} - \dfrac{xe^{2x}}{2x + 1}$.

Find the derivative and sketch the curves of the functions below. Give the domain of each of the functions. List all maxima and minima for each graph. Also, give the x and y-intercepts and any asymptotes if they exist.

5. $y = \dfrac{x^2}{x + 1}$,

6. $y = \dfrac{e^x}{x + 1}$,

7. $y = \dfrac{x^2 - 2x + 2}{x - 1}$,

8. $y = \dfrac{x^2}{x^2 + 1}$,

9. Consider the chalone model for mitosis given by the equation

$$P_{n+1} = f(P_n) = \frac{2P_n}{1 + (0.05P_n)^2},$$

where P_n is the population density at the n^{th} time step.

a. Find all equilibria of the model, that is, what population densities remain constant for any time period.

b. Differentiate the updating function $f(P_n)$. Find the population density P_n that results in the highest mitotic increase in density at the next time step.

c. Sketch a graph of $f(P)$ for $P \geq 0$, showing the intercepts, all extrema, and any asymptotes.

10. Carbon monoxide (CO) binds about 200 times more effectively than oxygen to hemoglobin, forming the complex called carboxyhemoglobin. This tight binding affinity makes death by carbon monoxide poisoning a problem in industrial settings as well as from running cars in a garage. Let p by the partial pressure of CO measured in torrs, then a dissociation curve for CO and hemoglobin is given by

$$y(p) = \frac{p^4}{0.0625 + p^4},$$

where y is the fraction of hemoglobin bound by CO.

a. Differentiate $y(p)$ and also find the second derivative, $y''(p)$. Find the values of p that satisfy $y''(p) = 0$. Give the p and y values for any points of inflection ($p \geq 0$).

b. Find any intercepts and asymptotes for $y(p)$, then sketch a graph of $y(p)$. Find the partial pressure of CO that results in the hemoglobin being 90% saturated. Compare this dissociation curve to the one in the lecture notes.

c. For $y'(p)$, find any intercepts, asymptotes, and extrema, then sketch a graph of $y'(p)$. Relate the maximum here to the point of inflection found in Part a.

11. As noted in Example 3, Jacob and Monod developed the theory of genetic control by induction. This is a very important control made most famous by the *lac* operon. The enzyme β-galactosidase is induced to catalyze the break down of lactose into simple sugars (glucose and galactose) for energy in the cell. The rate of induction for β-galactosidase is given by the formula

$$R(L) = \frac{V_{max} L^2}{K + L^2},$$

where L is the concentration of lactose and V_{max} and K are kinetic constants.

a. Suppose that $V_{max} = 10$ and $K = 1$. Differentiate this rate function and also find its second derivative. Give both the L and R values for any points of inflection ($L \geq 0$).

b. Find any intercepts and asymptotes for $R(L)$, then sketch a graph of $R(L)$.

c. For $R'(L)$, find any intercepts, asymptotes, and extrema, then sketch the graph of $R'(L)$. Relate the maximum here to the point of inflection in Part a.

12. Some entomologists use Hassell's model for studying the population of insects. An updating function that gives the population of the insects in the next generation (or time period) is given by

$$H(P) = \frac{5P}{1 + 0.004P},$$

where P is the current population of insects.

a. Differentiate this function, then find the second derivative. Use the second derivative to find any points of inflection ($P \geq 0$), giving both the P and H values.

b. Find any intercepts and asymptotes for $H(P)$ ($P \geq 0$). Sketch a graph of $H(P)$.

c. At what population is $H(P)$ increasing most rapidly, and what is this rate of increase? Find the intercepts and asymptotes for $H'(P)$ ($P \geq 0$), then sketch the graph for this function.

13. The logistic growth model is one of the most common models used in ecological research. Consider a yeast population that satisfies the logistic growth model

$$Y(t) = \frac{1000}{1 + 19e^{-0.1t}},$$

where t is in hours and Y is in yeast/cc.

a. Differentiate this function, then find the second derivative. Use the second derivative to find any points of inflection ($t \geq 0$), giving both the t and Y values.

b. Find any intercepts and asymptotes for $Y(t)$ ($t \geq 0$). Sketch a graph of $Y(t)$. Find how long it takes for the initial population to double.

c. When is $Y(t)$ increasing most rapidly, and what is this rate of increase? Find the intercepts and asymptotes for $Y'(t)$ ($t \geq 0$), then sketch the graph for this function.

d. A Malthusian growth model that approximates this yeast population during the early stages of growth is given by

$$M(t) = 50e^{0.1t}.$$

Find a doubling time from this model, then compare your result to the logistic growth model above.