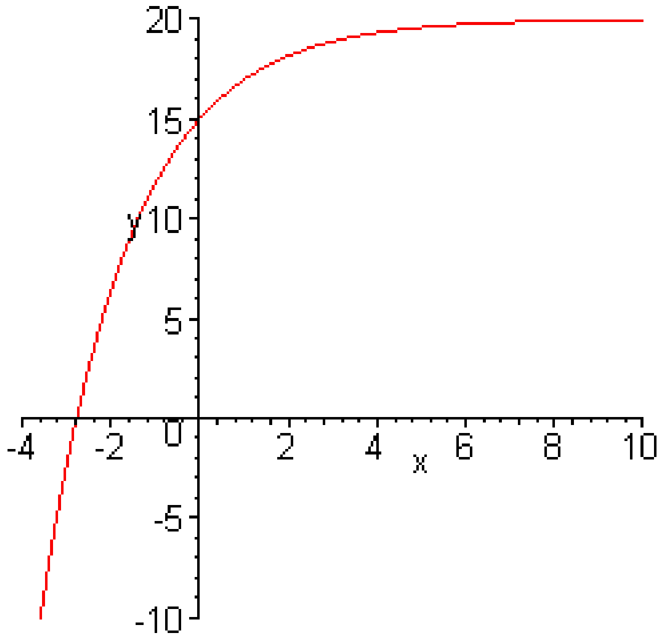
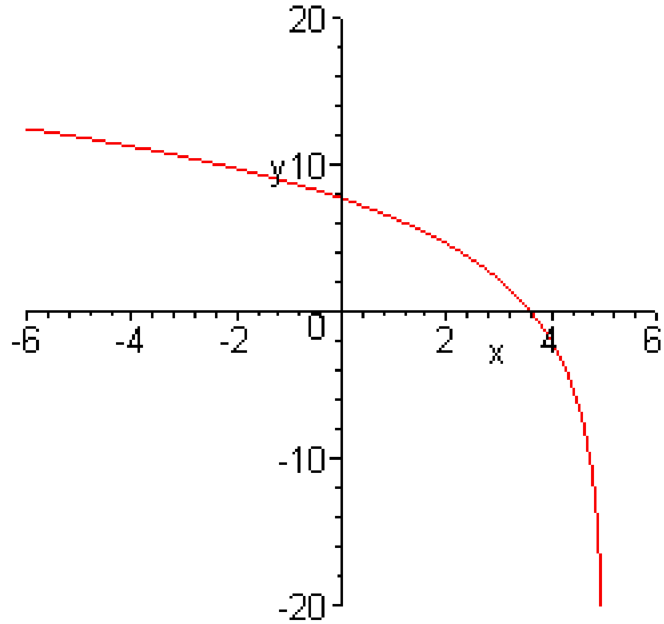
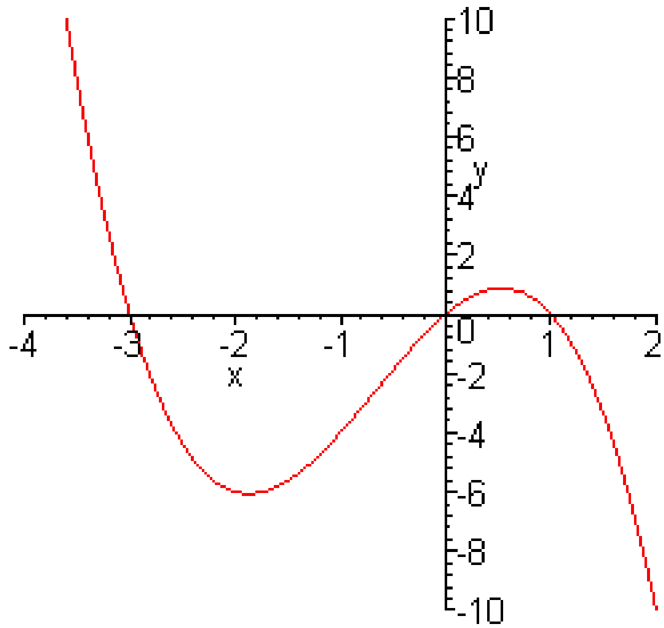
1. a. For $y = 20 - 5e^{-0.5x}$, the domain is all x. The y-intercept is where $x = 0$, so $y = 20 - 5e^{-0} = 15$. The x-intercept satisfies $y = 0$, so $20 - 5e^{-0.5x} = 0$ or $e^{-0.5x} = 4$, which gives $-0.5x = \ln(4)$ or $x = -2\ln(4) \simeq -2.773$. Thus, the intercepts are at $(-2.773, 0)$ and $(0, 15)$. There is a horizontal asymptote as $x \to \infty$ with $y = 20 - 5(0) = 20$. The graph is below to the left.



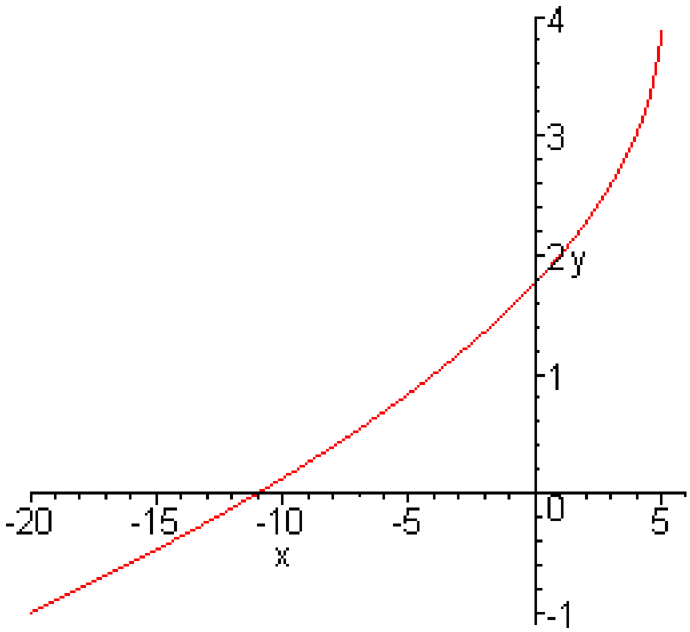


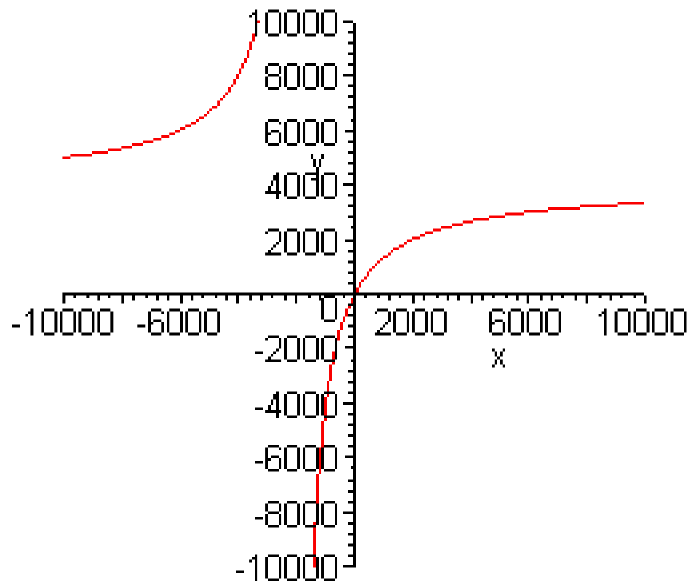
b. For $y = 6\ln(5 - x) - 2$, the domain satisfies $5 - x > 0$ or $x < 5$. The y-intercept is where $x = 0$, so $y = 6\ln(5) - 2 \simeq 7.657$. The x-intercept satisfies $y = 0$, so $6\ln(5 - x) - 2 = 0$ or $\ln(5 - x) = \frac{1}{3}$, which gives $5 - x = e^{1/3}$ or $x = 5 - e^{1/3} \simeq 3.604$. Thus, the intercepts are at $(3.604, 0)$ and $(0, 7.657)$. There is a vertical asymptote to the edge of the domain with $x = 5$ with $y \to -\infty$. The graph is above to the right.

c. For $y = 3x - 2x^2 - x^3$, the domain is all x. The y-intercept is where $x = 0$, so $y = 0$. The x-intercept satisfies $y = 0$, so $3x - 2x^2 - x^3 = -x(x^2 + 2x - 3) = -x(x + 3)(x - 1) = 0$ or $x = 0, -3, 1$. Thus, the intercepts are at $(-3, 0)$, $(0, 0)$ and $(1, 0)$. There are no asymptotes. The graph is below.



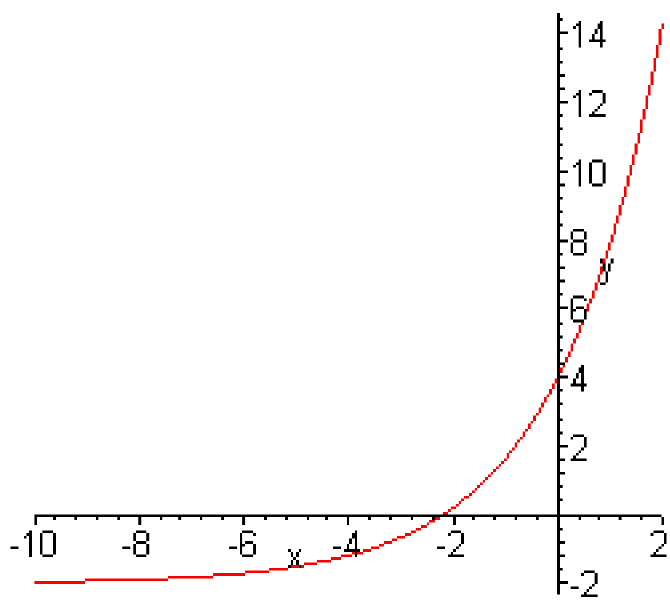
d. For $y = 4 - \sqrt{5-x}$, the domain satisfies $5 - x \geq 0$ or $x \leq 5$. The y-intercept is where $x = 0$, so $y = 4 - \sqrt{5} \simeq 1.7639$. The x-intercept satisfies $y = 0$, so $4 - \sqrt{5-x} = 0$ or $\sqrt{5-x} = 4$, which gives $5 - x = 16$ or $x = -11$. Thus, the intercepts are at $(1.7639, 0)$ and $(0, -11)$. There are no asymptotes. The graph is below to the left.



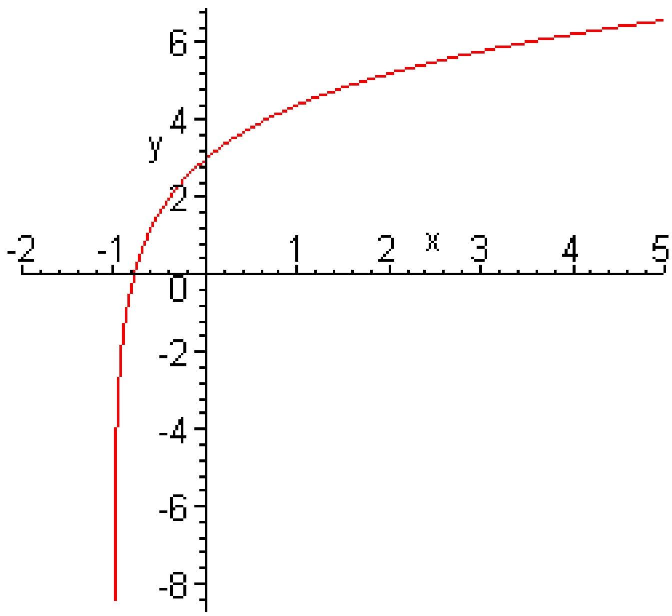


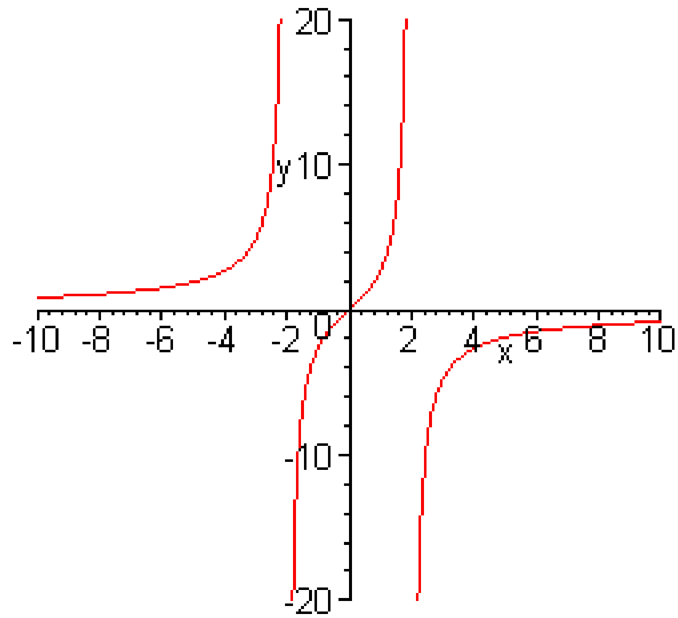
e. For $y = \dfrac{4x}{2 + 0.001x}$, the domain is all x except when $2 + 0.001x = 0$ or $x = -2000$, so domain is $x \neq -2000$. This gives a vertical asymptote at $x = -2000$. The x and y-intercept is $(0, 0)$. A horizontal asymptote is found by considering the highest powers in the numerator and denominator, so $y \to \dfrac{4x}{0.001x} = 4000$ The graph is above to the right.

f. For $y = 6e^{x/2} - 2$, the domain is all x. The y-intercept is where $x = 0$, so $y = 6e^{-0} - 2 = 4$. The x-intercept satisfies $y = 0$, so $6e^{x/2} - 2 = 0$ or $e^{x/2} = \frac{1}{3}$, which gives $\frac{x}{2} = \ln\left(\frac{1}{3}\right) = -\ln(3)$ or $x = -2\ln(3) \simeq -2.197$. Thus, the intercepts are at $(-2.197, 0)$ and $(0, 4)$. There is a horizontal asymptote as $x \to -\infty$ with $y = 6(0) - 2 = -2$. The graph is below.



g. For $y = 3 + 2\ln(x+1)$, the domain satisfies $x + 1 > 0$ or $x > -1$. The y-intercept is where $x = 0$, so $y = 3 + 2\ln(1) = 3$. The x-intercept satisfies $y = 0$, so $3 + 2\ln(x+1)$ or $\ln(x+1) = -\frac{3}{2}$, which gives $x + 1 = e^{-3/2}$ or $x = e^{-3/2} - 1 \simeq -0.7769$. Thus, the intercepts are at $(-0.7769, 0)$ and $(0, 3)$. There is a vertical asymptote to the edge of the domain with $x = -1$ with $y \to -\infty$. The graph is below to the left.





h. For $y = \dfrac{8x}{4 - x^2}$, the domain is all x except when $4 - x^2 = 0$ or $x = \pm 2$, so domain is $x \neq \pm 2$. This gives vertical asymptotes at $x = \pm 2$. The x and y-intercept is $(0, 0)$. A horizontal asymptote is found by considering the highest powers in the numerator and denominator, so $y \to \dfrac{8x}{-x^2} = \dfrac{8}{-x} \to 0$ as $x \to \infty$. The graph is above to the right.

2. a. Let $H_{n+1} = 1.02H_n$ with $H_0 = 2000$, then $H_1 = 1.02H_0 = 1.02(2000) = 2040$ and $H_2 = 1.02H_1 = 1.02(2040) = 2080.8$. In the general solution is given by $H_n = (1.02)^n H_0 = 2000(1.02)^n$.

b. For $G_{n+1} = 1.03G_n$ with $G_0 = 200$, the general solution is given by $G_n = (1.03)^n G_0 = 200(1.03)^n$. The doubling time for this population satisfies $2G_0 = (1.03)^n G_0$ or $(1.03)^n = 2$, which gives $n\ln(1.03) = \ln(2)$ or $n = \frac{\ln(2)}{\ln(1.03)} \simeq 23.45$. It takes 23.5 generations for the population to double.

c. The two populations are equal when $G_n = H_n$, so $200(1.03)^n = 2000(1.02)^n$ or $\left(\frac{1.03}{1.02}\right)^n = \frac{2000}{200} = 10$. Thus,

$$n = \frac{\ln(10)}{\ln\left(\frac{1.03}{1.02}\right)} \simeq 236.$$

The populations are equal in 236 generations.

3. a. The general solution for the Malthusian growth in the U. S. is $P_n = (1+r)^n P_0$, where $n = 0$ is 1960. Let $P_0 = 179.3$, then $P_{20} = 226.5$ It follows from the solution above that $226.5 = 179.3(1+r)^{20}$ or $(1+r)^{20} = \frac{226.5}{179.3}$. So $1+r = \frac{226.5}{179.3}^{\frac{1}{20}} = 1.011753$. The annual growth rate is $r = 0.011753$ (about 1.2% per year) and the general equation is $P_n = (1.011753)^n 179.3$.

b. The year 2000 is 40 years after 1960, so $P_{40} = 179.3(1.011753)^4 0 = 286.1$. Thus, the model predicts a population of 286.1 million in the year 2000. The error between this and the actual population is given by $\frac{|286.1-281.4|}{281.4} = 1.7\%$.

c. According to the model, $2P_0 = (1.011753)^n P_0$ or $(1.011753)^n = 2$. So $n \ln(1.011753) = \ln(2)$ or $n = \frac{\ln(2)}{\ln(1.011753)} = 59.32$. Thus, the population will double about 59.3 years after 1960, or about the year 2019.

4. a. The general solution for the Malthusian growth in France is $P_n = (1+r)^n P_0$, where $n = 0$ is 1980 and n is in decades. Let $P_0 = 53.9$, then $P_1 = 56.7$. It follows from the solution above that $56.7 = 53.9(1+r)$ or $(1+r) = \frac{56.7}{53.9} = 1.051948$, so $r = 0.051948$. The growth rate is $r = 0.051948$ (about 5.2% per decade) and the general equation is $P_n = (1.051948)^n 53.9$.

b. The years 2000 and 2020 correspond to the decades 2 and 4, respectively. $P_2 = 53.9(1.051948)^2 = 59.645$ and $P_4 = 53.9(1.051948)^4 = 66.003$ The population predictions are therefore 59.6 million in the year 2000 and 66 million in the year 2020. The error in the year 2000 is determined by $\frac{|59.4-59.6|}{59.4} = 0.34\%$

c. In Kenya, the Malthusian growth model is $P_n = (1+r)^n P_0$ with $P_0 = 16.7$, then $P_1 = 24.2$. It follows from the solution above that $24.2 = 16.7(1+r)$ or $(1+r) = \frac{24.2}{16.7} = 1.4491$, so $r = 0.4491$. The growth rate is $r = 0.4491$ (about 44.9% per decade) and the general equation is $P_n = (1.4491)^n 16.7$. The years 2000 and 2020 correspond to the decades 2 and 4, respectively. $P_2 = 16.7(1.4491)^2 = 35.068$ and $P_4 = 16.7(1.4491)^4 = 73.640$ The population predictions are therefore 35.1 million in the year 2000 and 73.6 million in the year 2020.

For the population to double, $2P_0 = (1.4491)^n P_0$ or $(1.4491)^n = 2$. So $n \ln(1.4491) = \ln(2)$ or $n = \frac{\ln(2)}{\ln(1.4491)} = 1.8686$. It takes 1.87 decades, or about 18.7 years, for the population of Kenya to double.

d. The populations become equal when $53.9(1.051948)^n = 16.7(1.4491)^n$ or $\left(\frac{1.4491}{1.051948}\right)^n = \frac{53.9}{16.7}$. Thus, $1.377541^n = 3.227545$, which becomes $n \ln(1.377541) = \ln(3.227545)$ or $n = \frac{\ln(3.227545)}{\ln(1.377541)} = 3.658$. The populations become equal in 3.66 decades, so the population of Kenya will first exceed that of France in 2017.

e. For France, the annualized growth satisfies $53.9(1+r)^{10} = 56.7$ or $(1+r)^{10} = \frac{56.7}{53.9} = 1.051948$. Thus, $1+r = \frac{56.7}{53.9}^{\frac{1}{10}} = 1.005077$. Similarly for Kenya, the annualized growth satisfies $16.7(1+r)^{10} = 24.2$ or $(1+r)^{10} = \frac{24.2}{16.7} = 1.4491$. Thus, $1+r = \frac{24.2}{16.7}^{\frac{1}{10}} = 1.037791$. It follows that the annual growth rate in France is $r = 0.00508$, while in Kenya it is $r = 0.0378$.

5. a. The population, P_n, in organisms/liter after n weeks satisfies a nonautonomous Malthusian growth model

$$P_{n+1} = (1 + k(t_n))P_n \quad \text{with} \quad P_0 = 5,000.$$

where $k(t) = 0.12 - 0.03t$. It follows that

$$\begin{aligned} P_1 &= 5000(1 + 0.12 - 0.03(0)) = 5600 \\ P_2 &= 5600(1 + 0.12 - 0.03(1)) = 6104 \\ P_3 &= 6104(1 + 0.12 - 0.03(2)) = 6470.2 \\ P_4 &= 6470(1 + 0.12 - 0.03(3)) = 6664.3 \\ P_5 &= 6664(1 + 0.12 - 0.03(4)) = 6664.3 \end{aligned}$$

b. The growth rate falls to zero when $k(t) = 0.12 - 0.03t = 0$ or $t = 4$ weeks. From above, we see $P_4 = 6664.3$ organisms/liter.

c. The theoretical extinction level is when $1 + k(t) = 1.12 - 0.03t = 0$ or $t = 37.3$ weeks. If the model is iterated until it falls below 1 crustacean per liter of water, then $P_26 = 0.647$ at which point the crustaceans will be effectively extinct.

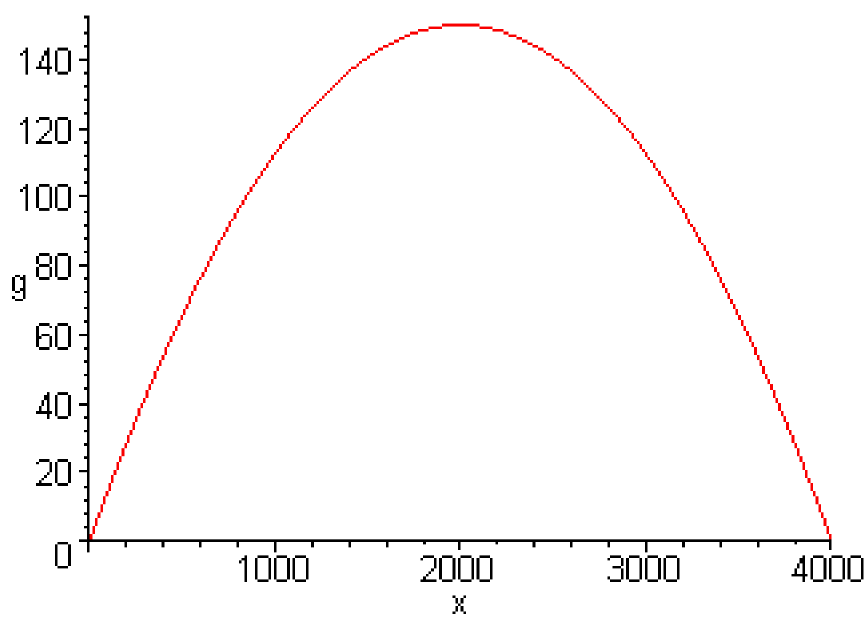
6. a. The discrete logistic growth model for yeast is given by

$$P_{n+1} = P_n + 0.15P\left(1 - \frac{P}{4000}\right).$$

Since $P_0 = 2000$, $P_1 = 2000 + 0.15(2000)\left(1 - \frac{2000}{4000}\right) = 2150$ and $P_2 = 2150 + 0.15(2150)\left(1 - \frac{2150}{4000}\right) = 2299$.

b. The growth rate, $g(P)$, is zero, when $0.15P\left(1 - \frac{P}{4000}\right) = 0$. This is in factored form, so either $P = 0$ or $\left(1 - \frac{P}{4000}\right) = 0$, which is equivalent to $P = 4000$. The maximum growth rate occurs half way between the two intercepts, so $P = 2000$. This maximum growth is $g(2000) = 0.15(2000)\left(1 - \frac{2000}{4000}\right) = 150$ individuals$\times 1000/\text{cm}^3$/hr. The sketch of $g(P)$ is below.

c. At equilibrium, $g(P) = 0$, so the two equilibria are $P = 0$ and $P = 4000$.



7. a. The population of herbivores satisfies the model $P_{n+1} = 1.1P_n$. Since $P_0 = 100$, then $P_1 = 1.1(100) = 110$ and $P_2 = 1.1(110) = 121$. The population doubles when $2P_0 = (1.1)^n P_0$ or $(1.1)^n = 2$. Thus, $n\ln(1.1) = \ln(2)$ or $n = \frac{\ln(2)}{\ln(1.1)} \simeq 7.2725$. It takes about 7.3 years for the population to double.

b. For the discrete logistic growth model, $P_{n+1} = 1.1P_n - 0.0005P_n^2$ with $P_0 = 100$, then $P_1 = 1.1(100) - 0.0005(100)^2 = 105$ and $P_2 = 1.1(105) - 0.0005(105)^2 = 109.9875$.

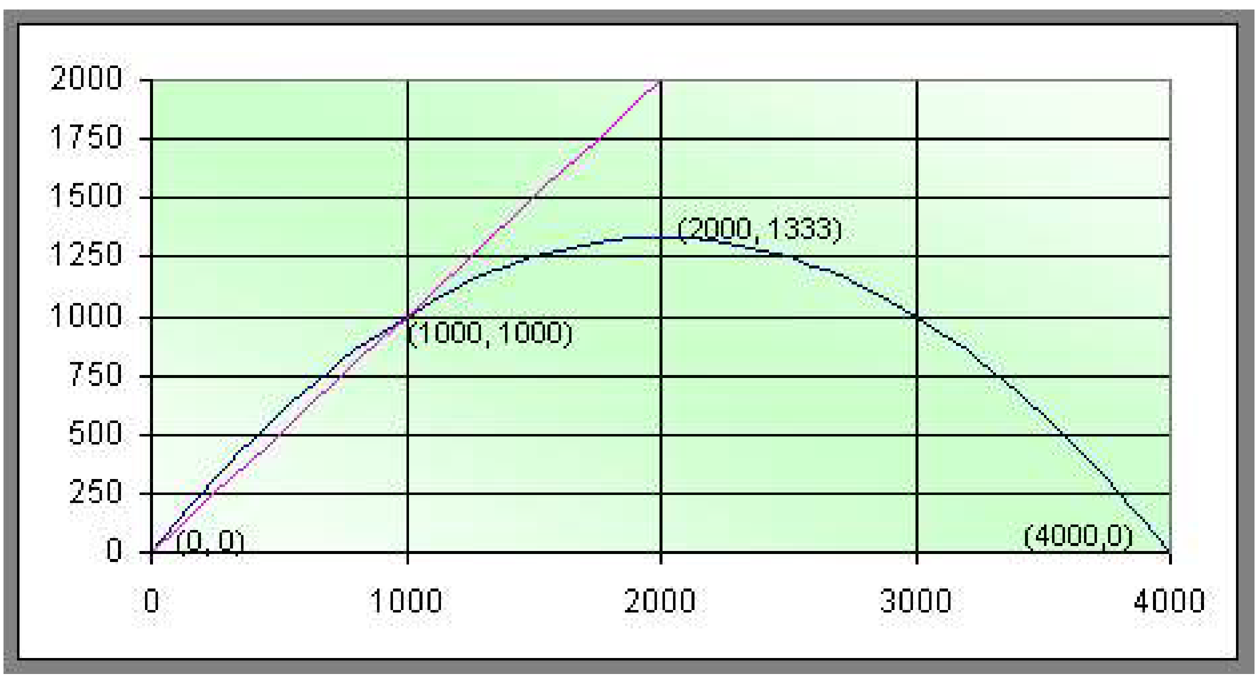
c. At equilibrium, $P_e = 1.1P_e - 0.0005P_e^2$ or $0.0005P_e^2 - 0.1P_e = 0.0005P_e(P_e - 200) = 0$. From this factored form, it follows that the equilibria are $P_e = 0$ or $P_e = 200$.

8. a. For the discrete logistic growth model, $p_{n+1} = \frac{4}{3}p_n - \frac{1}{3000}p_n^2$ with $p_0 = 100$, it follows that $p_1 = \frac{4}{3}(100) - \frac{1}{3000}(100)^2 = 130$ and $p_2 = \frac{4}{3}(130) - \frac{1}{3000}(130)^2 = 167.7$.

b. The p_n-intercepts satisfy $\frac{4}{3}p_n - \frac{1}{3000}p_n^2 = \frac{1}{3000}p_n(4000 - p_n) = 0$. Thus, $p_n = 0$ or 4000. The vertex is halfway between the intercepts, so $p_n = 2000$ and $f(2000) = \frac{4}{3}(2000) - \frac{1}{3000}(2000)^2 = 1333.3$. Thus, the vertex occurs at $(2000, 1333.3)$.

The function $f(p_n)$ intersects the identity map at the equilibria. The equilibria satisfy $p_e = \frac{4}{3}p_e - \frac{1}{3000}p_e^2$ or $\frac{1}{3000}p_e^2 - \frac{1}{3}p_e = \frac{1}{3000}p_e(p_e - 1000) = 0$, so $p_e = 0$ and 1000. Thus, the points of intersection are $(0, 0)$ and $(1000, 1000)$. The graph of the updating function and the identity map are shown below.

c. As seen above, the equilibria are $p_e = 0, 1000$. Based on the data in Part a., it appears that the equilibrium at $p_e = 0$ is unstable, and the one at $p_e = 1000$ is stable.



9. a. For the model with immigration, $p_{n+1} = 0.8p_n + 300$ with $p_0 = 500$, the next 3 populations are $p_1 = 0.8(500) + 300 = 700$, $p_2 = 0.8(700) + 300 = 860$, and $p_3 = 0.8(860) + 300 = 988$.

b. The equilibrium satisfies $p_e = 0.8p_e + 300$ or $0.2p_e = 300$. Thus, the equilibrium is $p_e = \frac{300}{0.2} = 1500$. The equilibrium is stable, because the iterations are moving towards it.

10. a. Since $c_{n+1} = (1-q)c_n + q\gamma$ with $\gamma = 5.2$ ppm, $c_0 = 50$ and $c_1 = 44.6$, we have $44.6 = (1-q)(50) + q(5.2)$ or $q(50-5.2) = 50 - 44.6$. Thus, $q = \frac{5.4}{44.8} = 0.120536$. The functional reserve capacity satisfies $V_r = (1-q)V_i/q(1-0.120536)\frac{300}{0.120536} = 2188.9$ ml.

b. The concentration of Helium in the next two breaths are

$$\begin{aligned} c_2 &= (0.879464)44.6 + (0.120536)5.2 = 39.85 \\ c_3 &= (0.879464)39.85 + (0.120536)5.2 = 35.67 \end{aligned}$$

As we saw in the class notes, $c_e = (1-q)c_e + q\gamma$ or $qc_e = q\gamma$, so $c_e = \gamma = 5.2$ ppm of He. This is a stable equilibrium, since the solution is approaching the equilibrium concentration.

11. a. Given the Malthusian growth model with dispersion, $P_{n+1} = (1 + r)P_n - \mu$, and the data $P_1 = 500$, $P_2 = 630$, and $P_3 = 825$, we find the constants r and μ by substitution and the simultaneous solution of two equations and two unknowns. We have

$$630 = 500(1 + r) - \mu \qquad \text{and} \qquad 825 = 630(1 + r) - \mu.$$

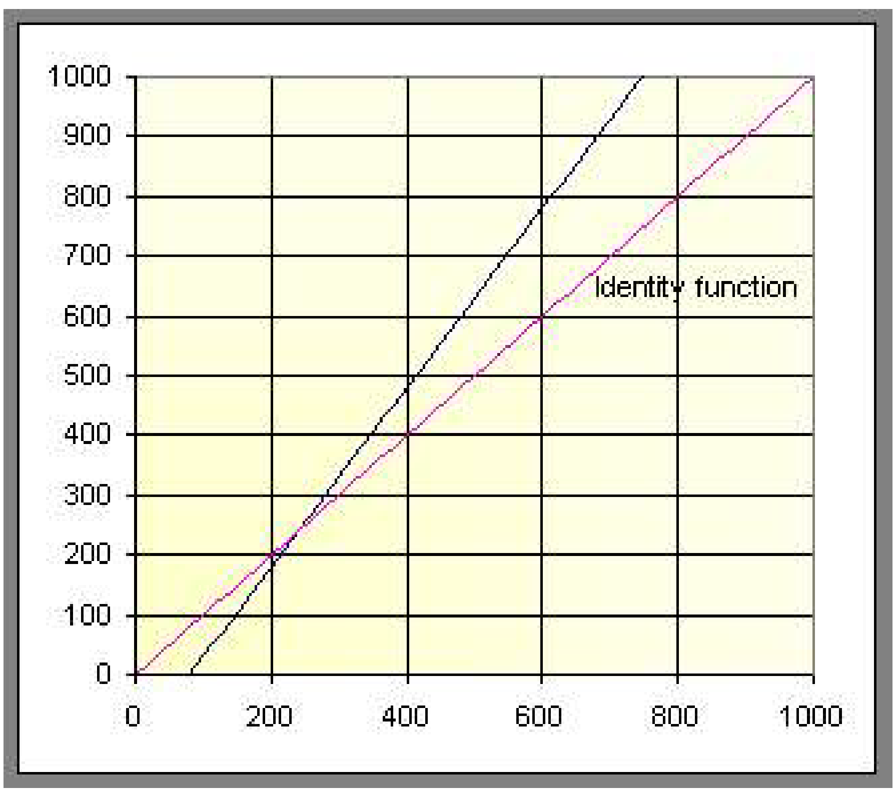
Subtracting the first equation from the second gives $825 - 630 = 630(1 + r) - 500(1 + r)$ or $130(1 + r) = 195$. It follows that $1 + r = \frac{195}{130} = 1.5$ or $r = 0.5$. This value is substituted into the first equation above to give $630 = 500(1.5) - \mu$, which gives $\mu = 120$.

Thus, the model becomes $P_{n+1} = 1.5P_n - 120$, and the next 2 weeks satisfy

$$\begin{aligned} P_3 &= (1.5)(825) - 120 = 1117.5 \\ P_4 &= (1.5)(1117.5) - 120 = 1556.25 \end{aligned}$$

b. At the equilibria, $P_e = 1.5P_e - 120$, so $0.5P_e = 120$ or $P_e = 240$. Since the population numbers are moving away from the equilibrium, the equilibrium is unstable.

c. The graph of the updating function and identity map, $P_{n+1} = P_n$, are shown below. The only point of intersection occurs at the equilibrium found above.

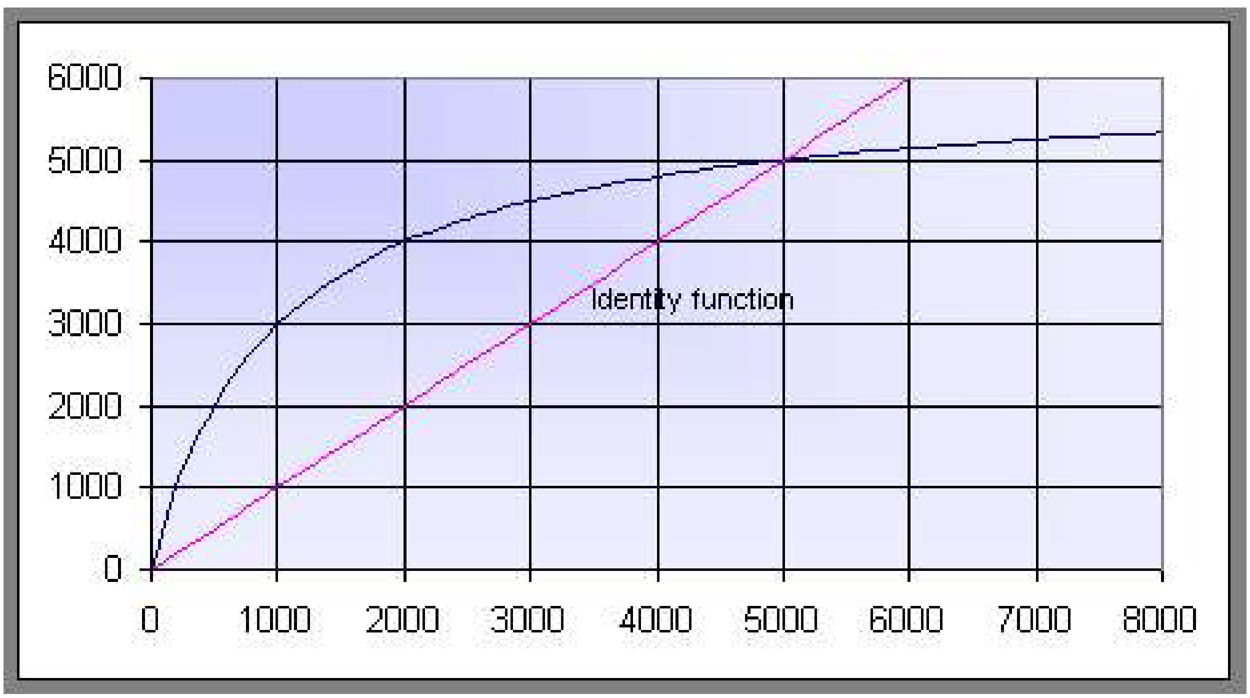


12. a. Hassell's model, $p_{n+1} = H(p_n) = \frac{6p_n}{1+0.001p_n}$, with $p_0 = 2000$ has the next two generations,

$$p_1 = \frac{6(2000)}{1+0.001(2000)} = 4000$$
$$p_2 = \frac{6(4000)}{1+0.001(4000)} = 4800$$

b. The equilibria for this model satisfy $p_e = \frac{6p_e}{1+0.001p_e}$ or $p_e(1+0.001p_e) = 6p_e$. Either $p_e = 0$ or $1+0.001p_e = 6$, which implies $0.001p_e = 5$ or $p_e = \frac{5}{0.001} = 5000$.

c. The only intercept is $p = 0$. The horizontal asymptote is found by examining the highest powers, so for large p, $H(p) \simeq \frac{6p}{0.001p} = \frac{6}{0.001} = 6000$. The graphs of $H(p)$ for $p > 0$ and the identity map, $p_{n+1} = p_n$ are shown below. These functions intersect at the equilibrium $(5000, 5000)$.



13. a. Since $a_{n+1} = 1.8a_n$ with $a_0 = 50,000$, the general solution is $a_n = 1.8^n a_0 = (50000)1.8^n$. After 5 hours, $a_5 = (50,000)1.8^5 = 944,784$.

b. From the selection model, $p_{n+1} = \frac{1.8p_n}{1.3+0.5p_n}$, with $p_1 = 0.5$, it is easy to compute p_2,

$$p_2 = \frac{1.8(0.5)}{1.3 + 0.5(0.5)} = 0.5806.$$

For p_0, we have $p_1 = \frac{1.8p_0}{1.3+0.5p_0}$ or $0.5 = \frac{1.8p_0}{1.3+0.5p_0}$, so $0.5(1.3 + 0.5p_0) = 1.8p_0$. Thus, $1.55p_0 = 0.65$, which gives $p_0 = 0.419355$.

c. At equilibrium, $p_e = \frac{1.8p_e}{1.3+0.5p_e}$ or $p_e(1.3 + 0.5p_e) = 1.8p_e$. This gives $0.5p_e(p_e - 1) = 0$. Thus, the equilibria are $p_e = 0$ and 1. Since $p_0 = 0.419$, $p_1 = 0.5$, and $p_2 = 0.581$, the solution is increasing towards the larger equilibrium. Thus, the limiting fraction for large n will be $p_e = 1$.

d. Below is the a graph of the updating function for the fraction of bacteria, p_n, of type a and the identity map for $0 \leq p \leq 1$.