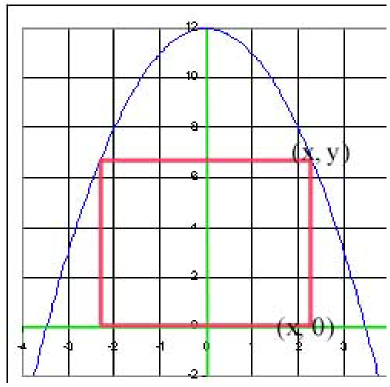
1. The parabola is symmetric about the y-axis and is show in the diagram below. The point



(x, y) in the diagram lies on the parabola and appears in the upper right corner of the rectangle. By the symmetry, we see that the area of the rectangle is $A = 2xy$, where y satisfies $y = 12 - x^2$. It follows that we can write

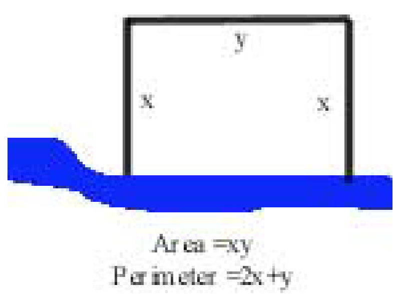
$$A(x) = 2x(12 - x^2) = 24x - 2x^3.$$

Differentiating this expression gives

$$A'(x) = 24 - 6x^2.$$

The maximum occurs when the derivative is zero, so $24 - 6x^2 = 0$ or $x^2 = 4$. Thus, $x = 2$ and $y = 12 - 2^2 = 8$. From the diagram, it follows that the dimensions of the largest rectangle inscribed in the parabola has a width of 4 and a height of 8. ($-2 \leq x \leq 2$, $0 \leq y \leq 8$) This gives the maximum area as $A_{max} = 4 \cdot 8 = 32$.

2. Begin this problem by drawing a diagram as shown below. The area of the study plot is $A = xy$ with the fence enclosing the region having length $P = 2x + y$, where x is perpendicular to the river and y is parallel to the river.



Since there is 20 m of fence, $2x + y = 20$ or $y = 20 - 2x$. The area of the region satisfies

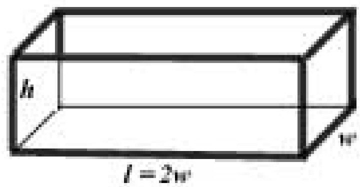
$$A(x) = x(20 - 2x) = 20x - 2x^2.$$

Differentiating $A(x)$ yields

$$A'(x) = 20 - 4x,$$

which is zero when $20 - 4x = 0$ or $x = 5$. It follows that $y = 20 - 2(5) = 10$. Thus, the maximum study area has 5 m of fence perpendicular to the river and 10 m of fence parallel to the river with a maximum area of $A_{max} = 50$ m^2.

3. Below is a diagram of the rectangular box described in the problem.



The volume of the box is given by $V = lwh = (2w)wh$. The surface area of the open box satisfies

$$S = lw + 2lh + 2wh = 2w^2 + 4wh + 2wh = 2w^2 + 6wh = 600\text{in}^2.$$

This equation is solved giving $wh = \frac{600-2w^2}{6} = 100 - \frac{w^2}{3}$ or $h = \frac{100}{w} - \frac{w}{3}$. This is substituted into the equation for the volume of the box resulting in

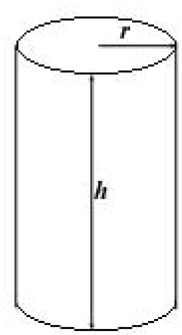
$$V(w) = 2w\left(100 - \frac{w^2}{3}\right) = 200w - \frac{2}{3}w^3.$$

Differentiating $V(w)$ gives

$$V'(w) = 200 - 2w^2.$$

Setting the derivative equal to zero gives $2w^2 = 200$ or $w^2 = 100$, so $w = 10$ in (the width of the box). Thus, the length is $l = 2w = 20$ in. The height is $h = \frac{100}{10} - \frac{10}{3} = \frac{20}{3}$ in. The maximum volume is $V_{max} = 20 \cdot 10 \cdot \frac{20}{3} = 4000/3$ in^3.

5. To the right is a diagram of the can for this problem



The volume of the can satisfies

$$V = \pi r^2 h = 1000 \text{ cm}^3.$$

The surface area of the can is given by

$$S = 2\pi rh + 2\pi r^2 \text{ cm}^2,$$

since it consists of the lateral side and two circular ends. The condition on the volume gives the height

$$h = \frac{1000}{\pi r^2}.$$

Substituting this expression into the equation for the surface area gives

$$S(r) = 2\pi r \frac{1000}{\pi r^2} + 2\pi r^2 = 2000r^{-1} + 2\pi r^2.$$

Differentiating this expression yields

$$S'(r) = -2000r^{-2} + 4\pi r = \frac{4\pi r^3 - 2000}{r^2}.$$

The derivative is zero at the minimum when the numerator of the expression above is zero, so

$$4\pi r^3 - 2000 = 0 \qquad \text{or} \qquad r^3 = \frac{1000}{2\pi}.$$

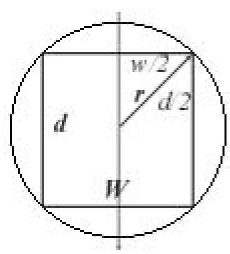
By taking the cube root of both sides we have the radius of the can is

$$r = \frac{10}{\sqrt[3]{2\pi}} \simeq 5.419 \text{ cm}.$$

The height satisfies

$$h = \frac{1000}{\pi\left(\frac{10}{\sqrt[3]{2\pi}}\right)^2} = 10\sqrt[3]{\frac{4}{\pi}} \simeq 10.839 \text{ cm}.$$

6. The beam with width w and depth d that is cut from a circular log with a radius of r is shown in the diagram below. It satisfies the equation



$$\left(\frac{w}{2}\right)^2 + \left(\frac{d}{2}\right)^2 = r^2 \qquad \text{or} \qquad w^2 + d^2 = 4r^2.$$

With the constant k for the proportionality constant, then the strength of the beam is $S = kwd^2$. From the expression above, $d^2 = 4r^2 - w^2$, so

$$S(w) = kw(4r^2 - w^2) = k(4r^2w - w^3).$$

To find the strongest beam, we differentiate $S(w)$, then set the derivative equal to zero. Thus,

$$S'(w) = k(4r^2 - 3w^2) = 0.$$

So the width of the beam satisfies

$$3w^2 = 4r^2 \qquad \text{or} \qquad w = \frac{2r}{\sqrt{3}},$$

and the depth of the beam is given by

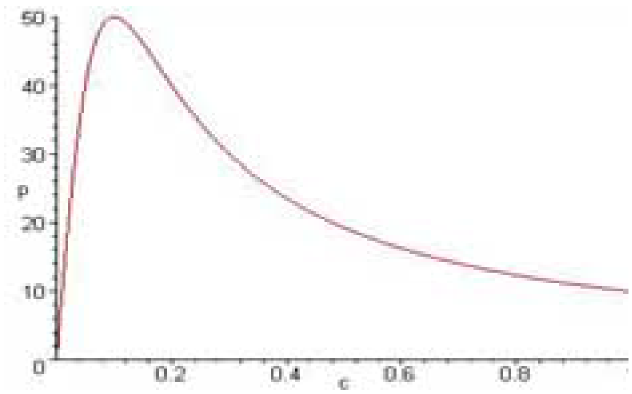
$$d^2 = 4r^2 - \frac{4}{3}r^2 = \frac{8}{3}r^2 \qquad \text{or} \qquad d = 2r\sqrt{2/3}.$$

8. a. The maximum value of $P(c)$ occurs when the derivative of P with respect to c is zero. We use the quotient rule, so

$$P'(c) = \frac{1000(1 + 100c^2) - 1000c(200c)}{(1 + 100c^2)^2} = \frac{1000(1 - 100c^2)}{(1 + 100c^2)^2}.$$

This is zero when the numerator is zero, so $1000(1 - 100c^2) = 0$ or $100c^2 = 1$. This gives $c^2 = \frac{1}{100}$ or $c = 0.1$. Since $P(0.1) = \frac{1000(0.1)}{1+100(0.1)^2} = 50$, we have that the optimal concentration is $c = 0.1$ M, and the maximal population density is $P_{max} = 50$ organisms/cm^2.

b. Note the graph passes through the origin and has a horizontal asymptote of $P = 0$ for large values of c (as described by both the problem and shown by the function. Below is the graph of this function



10. a. The model for age-structured populations is given by

$$r(x) = \frac{\ln(e^{-ax}bx^c)}{x} = \frac{\ln(e^{-ax}) + \ln(b) + c\ln(x)}{x} = \frac{-ax + \ln(b) + c\ln(x)}{x}.$$

From the quotient rule, the derivative is given by

$$r'(x) = \frac{x(-a + c/x) - (-ax + \ln(b) + c\ln(x))}{x^2} = \frac{c - \ln(b) - c\ln(x)}{x^2}.$$

The optimal rate of increase is assumed to be the maximum rate. Setting the derivative equal to zero, we take only the numerator from the derivative above. Thus,

$$c - \ln(b) - c\ln(x) = 0 \qquad \text{or} \qquad \ln(x) = 1 - \frac{\ln(b)}{c}.$$

Exponentiating

$$x = exp\left(1 - \frac{\ln(b)}{c}\right) = e \cdot b^{-1/c}.$$

Thus, the optimal age of reproduction is $x = e \cdot b^{-1/c}$. For parameters in Part b, this is $x = 0.582$.

b. For the graphs, see the short solutions.