1. (6pts) Consider the transcendental equation with a small nonlinear perturbation:

$$x^2 - x - 6 = \varepsilon \cos(x), \qquad \text{with} \quad \varepsilon \ll 1.$$

We let

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \mathcal{O}\left(\varepsilon^3\right),$$

then

$$\cos(x) = \cos(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \mathcal{O}\left(\varepsilon^3\right)) = \cos(x_0) - \varepsilon x_1 \sin(x_0) + \mathcal{O}\left(\varepsilon^2\right).$$

The equation above becomes:

$$\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \mathcal{O}\left(\varepsilon^3\right)\right)^2 - \left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \mathcal{O}\left(\varepsilon^3\right)\right) - 6 = \varepsilon\Big(\cos(x_0) - \varepsilon x_1 \sin(x_0) + \mathcal{O}\left(\varepsilon^2\right)\Big).$$

Expanding in orders of ε, we find:

$$x_0^2 - x_0 - 6 + \varepsilon\Big(2x_0x_1 - x_1 - \cos(x_0)\Big) + \varepsilon^2\Big(2x_0x_2 + x_1^2 - x_2 + x_1\sin(x_0)\Big) + \mathcal{O}\left(\varepsilon^3\right) = 0.$$

The two roots are perturbations of the roots, $x_0 = -2$ and 3. Sequentially, we have:

$$\begin{aligned}
\varepsilon^0 &: \qquad x_0^2 - x_0 - 6 = (x_0+2)(x_0-3) = 0 \quad \text{or} \quad x_0 = -2, 3,\\
\varepsilon^1 &: \qquad 2x_0x_1 - x_1 = \cos(x_0) \quad \text{or} \quad x_1 = \frac{\cos(x_0)}{2x_0 - 1},\\
\varepsilon^2 &: \qquad 2x_0x_2 + x_1^2 - x_2 = -x_1\sin(x_0) \quad \text{or} \quad x_2 = -\frac{x_1(\sin(x_0) + x_1)}{2x_0 - 1}.
\end{aligned}$$

For $x_0 = -2$, we have:

$$x_1 = -\frac{\cos(2)}{5} \approx 0.083229367, \qquad x_2 = \frac{x_1(x1 - \sin(2))}{5} \approx -0.013750624.$$

For $x_0 = 3$, we have:

$$x_1 = \frac{\cos(3)}{5} \approx -0.197998499, \qquad x_2 = -\frac{x_1(\sin(3) + x_1)}{5} \approx -0.002252371.$$

At $x_0 = -2$, MatLab gives the roots of our equation as $x = -1.991812826$, when $\varepsilon = 0.1$ and $x = -1.999169080$, when $\varepsilon = 0.01$. The two and three term approximations are:

$$\begin{aligned}
x &= -2 + 0.1x_1 = -1.991677063, & x &= -2 + 0.1x_1 + 0.01x_2 = -1.991814570,\\
x &= -2 + 0.01x_1 = -1.999167706, & x &= -2 + 0.01x_1 + 0.0001x_2 = -1.999169081.
\end{aligned}$$

For $\varepsilon = 0.1$, the approximations have 4 and 6 significant figures, while for $\varepsilon = 0.01$, the approximations have 6 and 9 significant figures, respectively.
At $x_0 = 3$, MatLab gives the roots of our equation as $x = 2.980181417$, when $\varepsilon = 0.1$ and $x = 2.998019794$, when $\varepsilon = 0.01$. The two and three term approximations are:

$$\begin{aligned}
x &= 3 + 0.1x_1 = 2.980200150, & x &= 3 + 0.1x_1 + 0.01x_2 = 2.980177626,\\
x &= 3 + 0.01x_1 = 2.998020015, & x &= 3 + 0.01x_1 + 0.0001x_2 = 2.998019790.
\end{aligned}$$

For $\varepsilon = 0.1$, the approximations again have 4 and 6 significant figures, while for $\varepsilon = 0.01$, the approximations again have 6 and 9 significant figures, respectively.

2. (5pts) a. The Bernoulli's IVP given by:

$$\frac{dy}{dt} + y = \varepsilon y^3, \qquad \text{with} \quad y(0) = 1,$$

is solved using the change of variables $u = y^{1-3} = y^{-2}$, so $\frac{du}{dt} = -2y^{-3}\frac{dy}{dt}$. Multiplying the ODE above by $-2y^{-3}$ gives:

$$-2y^{-3}\frac{dy}{dt} - 2y^{-2} = -2\varepsilon \qquad \text{or} \qquad \frac{du}{dt} - 2u = -2\varepsilon.$$

This is a linear ODE with integrating factor $\mu(t) = e^{-2t}$, so

$$\frac{d}{dt}\left(e^{-2t}u\right) = -2\varepsilon e^{-2t}, \qquad \text{or} \qquad u(t) = \varepsilon + ce^{2t}.$$

Since $y(0) = 1$, then $u(0) = 1$. It follows that $c = 1 - \varepsilon$, so

$$u(t) = \varepsilon(1 - e^{2t}) + e^{2t} = y^{-2}(t) \qquad \text{or} \qquad y^2(t) = \frac{e^{-2t}}{1 + \varepsilon(e^{-2t} - 1)}.$$

Taking the positive square root, we find:

$$y(t) = \frac{e^{-t}}{\sqrt{1 + \varepsilon(e^{-2t} - 1)}} = e^{-t}\left(1 + \varepsilon(e^{-2t} - 1)\right)^{-\frac{1}{2}}.$$

This is readily expanded using the p-series:

$$\begin{aligned} y(t) &= e^{-t}\left(1 - \frac{1}{2}\varepsilon(e^{-2t} - 1) + \frac{3/4}{2!}\varepsilon^2(e^{-2t} - 1)^2 + \mathcal{O}\left(\varepsilon^3\right)\right), \\ &= e^{-t} - \frac{\varepsilon}{2}(e^{-3t} - e^{-t}) + \frac{3\varepsilon^2}{8}(e^{-5t} - 2e^{-3t} + e^{-t}) + \mathcal{O}\left(\varepsilon^3\right). \end{aligned}$$

b. (5pts) Assuming a solution in the form:

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \mathcal{O}\left(\varepsilon^3\right),$$

with the initial conditions:

$$y_0(0) = 1, \qquad y_1(0) = y_2(0) = \cdots = 0,$$

we substitute into the Bernoulli's ODE above and obtain:

$$y_0'(t) + \varepsilon y_1'(t) + \varepsilon^2 y_2'(t) + \mathcal{O}\left(\varepsilon^3\right) + \left(y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \mathcal{O}\left(\varepsilon^3\right)\right) = \varepsilon\left(y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \mathcal{O}\left(\varepsilon^3\right)\right)^3.$$

We solve the problems successively for the different powers of ε.

$$\varepsilon^0: \qquad y_0' + y_0 = 0 \quad \text{with} \quad y_0 = 1,$$

gives $y_0(t) = e^{-t}$. Next

$$\varepsilon^1: \qquad y'_1 + y_1 = y_0^3 = e^{-3t} \quad \text{with} \quad y_1 = 0,$$

has the integrating factor $\mu(t) = e^t$. Thus,

$$\frac{d}{dt}\left(e^t y_1\right) = e^{-2t} \qquad \text{or} \qquad y_1(t) = -\frac{1}{2}e^{-3t} + ce^{-t}.$$

With the IC we have $y_1(t) = \frac{1}{2}(e^{-t} - e^{-3t})$. Next

$$\varepsilon^2: \qquad y'_2 + y_2 = 3y_0^2 y_1 = \frac{3}{2}(e^{-3t} - e^{-5t}) \quad \text{with} \quad y_2 = 0,$$

has the integrating factor $\mu(t) = e^t$. Thus,

$$\frac{d}{dt}\left(e^t y_2\right) = \frac{3}{2}(e^{-2t} - e^{-4t}) \qquad \text{or} \qquad y_2(t) = -\frac{3}{8}(2e^{-3t} - e^{-5t}) + ce^{-t}.$$

With the IC we have $y_2(t) = \frac{3}{8}(e^{-t} - 2e^{-3t} + e^{-5t})$. We combine these results to obtain:

$$y(t) = e^{-t} + \frac{\varepsilon}{2}(e^{-t} - e^{-3t}) + \frac{3\varepsilon^2}{8}(e^{-t} - 2e^{-3t} + e^{-5t}) + \mathcal{O}\left(\varepsilon^3\right).$$

These terms are readily seen to match the ones in the power series of Part a.