1. (4pts) In lecture we showed that a 2^{nd} order Cauchy-Euler problem with repeated roots, $r = r_1$ had the general solution:

$$y(t) = t^{r_1}\big(c_1 + c_2\ln(t)\big).$$

Now consider the 3^{rd} order differential operator L by

$$L[y] = t^3y''' + \alpha t^2y'' + \beta ty' + \gamma y = 0, \tag{1}$$

where the *auxiliary equation* has three equal roots, so $F(r) = (r - r_1)^3$. Following the same operation as seen in the lecture notes, we have

$$\frac{\partial}{\partial r}L[t^r] = \frac{\partial}{\partial r}[t^rF(r)] = (r - r_1)^3t^r\ln(t) + 3(r - r_1)^2t^r,$$

which is clearly zero when $r = r_1$. However,

$$\frac{\partial}{\partial r}L[t^r] = L\left[\frac{\partial}{\partial r}(t^r)\right] = L[t^r\ln(t)],$$

which must be zero when $r = r_1$, showing that $y_2(t) = t_1^r\ln(t)$ is a second linearly independent solution. Taking a second partial derivative gives:

$$\frac{\partial^2}{\partial r^2}L[t^r] = \frac{\partial^2}{\partial r^2}[t^rF(r)] = (r - r_1)^3t^r\big(\ln(t)\big)^2 + 6(r - r_1)^2t^r\ln(t) + 6(r - r_1)t^r,$$

which again must be zero when $r = r_1$. Since

$$\frac{\partial^2}{\partial r^2}L[t^r] = \frac{\partial}{\partial r}L[t^r\ln(t)] = L\left[\frac{\partial}{\partial r}(t^r)\ln(t)\right] = t^r\big(\ln(t)\big)^2,$$

which must be zero when $r = r_1$, showing that $y_3(t) = t_1^r\big(\ln(t)\big)^2$ is a third linearly independent solution. It follows that the general solution to (1) is given by:

$$y(t) = t^{r_1}\Big(c_1 + c_2\ln(t) + c_3\big(\ln(t)\big)^2\Big).$$

2. (4pts) With the Cauchy-Euler method of taking $y(t) = t^r$, the 3^{rd} order linear homogeneous ODE given by:

$$t^3y''' + 9t^2y'' + 19ty' + 8y = 0,$$

has the *auxiliary equation* given by:

$$F(r) = r(r - 1)(r - 2) + 9r(r - 1) + 19r + 8 = r^3 + 6r^2 + 12r + 8 = (r + 2)^3 = 0.$$

It follows that $r_1 = -2$ is a triple root of the auxiliary equation. From Problem 1, it follows that the general solution to this problem is given by:

$$y(t) = \frac{1}{t^2}\Big(c_1 + c_2\ln(t) + c_3\big(\ln(t)\big)^2\Big).$$

3. (4pts) **Reduction of Order** (Jean D'Alembert (1717-1783)): If $y_1(x)$ is known for the linear ODE:

$$y'' + p(x)y' + q(x)y = 0,$$

then attempt a solution of the form $y(x) = v(x)y_1(x)$ with $y_1(x) \neq 0$. Since $y_1(x)$ is a known solution to the original equation, it follows that $y_1'' + p(x)y_1' + q(x)y_1 = 0$. With $y(x) = v(x)y_1(x)$, then

$$y'(x) = v'(x)y_1(x) + v(x)y_1'(x) \quad \text{and} \quad y''(x) = 2v'(x)y_1'(x) + v''(x)y_1(x) + v(x)y_1''(x).$$

These are substituted into the original equation, so

$$2v'(x)y_1'(x) + v''(x)y_1(x) + v(x)y_1''(x) + p(x)v'(x)y_1(x) + p(x)v(x)y_1'(x) + q(x)v(x)y_1(x) = 0.$$

This reduces to

$$v''(x)y_1(x) + v'\left[p(x)y_1(x) + 2y_1'(x)\right] = 0.$$

Now if we let $w(x) = v'(x)$, then:

$$y_1(x)w'(x) + w(x)\left(p(x)y_1(x) + 2y_1'(x)\right) = 0 \qquad \text{or} \qquad w' = -\left(p(x) + 2\frac{y_1'(x)}{y_1(x)}\right)w.$$

which is a linear 1^{st} order ODE in w. Separate w and take the integral of both sides

$$\ln(w) = \ln(v') = -2\ln(y_1) - \int p(x)\,dx.$$

Exponentiating we have:

$$\frac{dv}{dx} = e^{-2\ln(y_1) - \int p(x)\,dx} = \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}.$$

It follows by integrating both sides of the equation that:

$$v(x) = \int \frac{e^{-\int p(x)\,dx}}{[y_1(x)]^2}dx.$$

Thus, the second linearly independent solution satisfies:

$$y_2(x) = y(x) = v(x)\cdot y_1(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{[y_1(x)]^2}dx.$$

4. (4pts) a. Consider the following ODE:

$$xy'' + (1-2x)y' + (x-1)y = 0. \tag{2}$$

Show that $y_1(x) = e^x$ is a solution to this differential equation.

For $y_1(x) = e^x$, we have $y_1'(x) = e^x = y_1''(x)$. Substituting into the original equation:

$$xe^x + (1-2x)e^x + (x-1)e^x = e^x(x + x - 2x + 1 - 1) = 0,$$

so $y_1(x)$ is a solution.

b. Since $y_1(x) = e^x$ is one solution to (2), we use the **Reduction of Order** method to find $y_2(x)$ for (2). It follows that:

$$y_2(x) = e^x \int \frac{e^{\int (-\frac{1}{x}+2)dx}}{e^{2x}} dx = e^x \int \frac{x^{-1}e^{2x}}{e^{2x}} dx = e^x \ln|x|.$$

We can show that these solutions make a fundamental set of solutions by showing that the Wronskian of the two are nonzero.

$$W_{[y_1,y_2]} = \begin{vmatrix} e^x & e^x \ln(x) \\ e^x & e^x \ln(x) + \frac{e^x}{x} \end{vmatrix} = \frac{e^{2x}}{x}.$$

We can see that $W_{[y_1,y_2]} \neq 0$ for all x, thus making y_1, y_2 a fundamental set of solutions.