1. (3pts) The homogeneous ODE:

$$\dot{\mathbf{x}} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \mathbf{x},$$

has its matrix in Jordan form. It is easy to see that the eigenvalues are $\lambda = -2, -1 \pm i$. It follows from the lecture notes that the fundamental solution to this ODE is easily written:

$$\Phi(t) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-t}\cos(t) & e^{-t}\sin(t) \\ 0 & -e^{-t}\sin(t) & e^{-t}\cos(t) \end{pmatrix}$$

From the Corollary of Abel's formula, we show that $|\Phi(t)| \neq 0$

$$\begin{aligned} \begin{vmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-t}\cos(t) & e^{-t}\sin(t) \\ 0 & -e^{-t}\sin(t) & e^{-t}\cos(t) \end{vmatrix} &= e^{-2t} \begin{vmatrix} e^{-t}\cos(t) & e^{-t}\sin(t) \\ -e^{-t}\sin(t) & e^{-t}\cos(t) \end{vmatrix} \\ &= e^{-2t}\left(e^{-2t}\cos^2(t) + e^{-2t}\sin^2(t)\right) = e^{-4t}(1) \neq 0. \end{aligned}$$

Thus, we have a fundamental solution, $\Phi(t)$.

2. (5pts) From the lecture notes, we have the variation of parameters formula:

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)\ ds.$$

The nonhomogeneous ODE:

$$\dot{\mathbf{x}} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ 1 \\ t \end{pmatrix}, \qquad x(0) = \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \end{pmatrix},$$

has the fundamental solution, $\Phi(t)$, from above. It is easily seen that:

$$g(t) = \begin{pmatrix} e^{-2t} \\ 1 \\ t \end{pmatrix} \qquad \text{and} \qquad \Phi^{-1}(t) = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t}\cos(t) & -e^{t}\sin(t) \\ 0 & e^{t}\sin(t) & e^{t}\cos(t) \end{pmatrix}.$$

We begin by finding the particular solution with the help of Maple to solve the integrals. First we compute the integral in the formula:

$$\int_0^t \Phi^{-1}(s)g(s)\ ds = \int_0^t \begin{pmatrix} 1 \\ e^s(-s\sin(s) + \cos(s)) \\ e^s(s\cos(s) + \sin(s)) \end{pmatrix} ds = \begin{pmatrix} t \\ \frac{e^t}{2}\left(t\cos(t) - t\sin(t) + \sin(t)\right) \\ \frac{1}{2}\left(1 + e^t\left(t\cos(t) + t\sin(t) - \cos(t)\right)\right) \end{pmatrix}.$$

Again with the help of Maple, we multiply this result by $\Phi(t)$ to obtain the particular solution to this nonhomogeneous ODE, giving the following:

$$\mathbf{x}_p(t) = \begin{pmatrix} te^{-2t} \\ \frac{t}{2} + \frac{e^{-t}\sin(t)}{2} \\ \frac{t}{2} - \frac{1}{2} + \frac{e^{-t}\cos(t)}{2} \end{pmatrix}.$$

Since $\Phi^{-1}(0) = I$, the solution is given by $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \mathbf{x}_p(t)$. It follows that our solution is given by:

$$\mathbf{x}(t) = \begin{pmatrix} x_{10}e^{-2t} \\ x_{20}e^{-t}\cos(t) + x_{30}e^{-t}\sin(t) \\ -x_{20}e^{-t}\sin(t) + x_{30}e^{-t}\cos(t) \end{pmatrix} + \begin{pmatrix} te^{-2t} \\ \frac{t}{2} + \frac{e^{-t}\sin(t)}{2} \\ \frac{t}{2} - \frac{1}{2} + \frac{e^{-t}\cos(t)}{2} \end{pmatrix}.$$

3. (3pts) The homogeneous ODE:

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 2t^{-2} & -2t^{-1} \end{pmatrix}\mathbf{x}, \qquad t > 0,$$

is transformed into a Cauchy-Euler equation by letting $y(t) = x_1(t)$. The system above shows $y' = \dot{x}_1 = x_2$, so $y'' = \dot{x}_2 = 2t^{-2}x_1 - 2t^{-1}x_2 = 2t^{-2}y - 2t^{-1}y'$, so

$$t^2y'' + 2ty' - 2y = 0.$$

Trying $y(t) = t^r$ in the ODE above gives:

$$t^r(r(r-1) + 2r - 2) = t^r(r^2 + r - 2) = 0,$$

which gives the *auxiliary equation* $(r+2)(r-1) = 0$. Thus, the Cauchy-Euler equation has the general solution:

$$y(t) = c_1\frac{1}{t^2} + c_2t,$$

giving two linearly independent solutions, $y_1(t) = \frac{1}{t^2}$ and $y_2(t) = t$. We create a fundamental solution by letting the first row be these solutions and the second row being their derivatives. Thus, we take

$$\Phi(t) = \begin{pmatrix} \frac{1}{t^2} & t \\ -\frac{2}{t^3} & 1 \end{pmatrix}.$$

From the Corollary of Abel's formula, we compute det $|\Phi(t)|$, so

$$\begin{vmatrix} \frac{1}{t^2} & t \\ -\frac{2}{t^3} & 1 \end{vmatrix} = \frac{1}{t^2} + \frac{2}{t^2} = \frac{3}{t^2} \neq 0.$$

It follows that this is a fundamental solution.

4. (5pts) From the lecture notes, we have the variation of parameters formula:

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)g(s)\,ds.$$

The nonhomogeneous ODE:

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 2t^{-2} & -2t^{-1} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6t \\ 9t^{-4} \end{pmatrix}, \qquad x(1) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}, \qquad t > 0,$$

has the fundamental solution, $\Phi(t)$, from above. It is easily seen that:

$$g(t) = \begin{pmatrix} 6t \\ 9t^{-4} \end{pmatrix} \qquad \text{and} \qquad \Phi^{-1}(t) = \begin{pmatrix} \frac{t^2}{3} & -\frac{t^3}{3} \\ \frac{2}{3t} & \frac{1}{3} \end{pmatrix}.$$

We begin by finding the particular solution to solve the integrals. First we compute the integral in the formula:

$$\int_1^t \Phi^{-1}(s)g(s)\ ds = \int_1^t \begin{pmatrix} 2s^3 - \frac{3}{s} \\ 4 + \frac{3}{s^4} \end{pmatrix} ds = \begin{pmatrix} \frac{t^4}{2} - \frac{1}{2} - 3\ln(t) \\ 4t - 3 - \frac{1}{t^3} \end{pmatrix}.$$

With the help of Maple we multiply this result by $\Phi(t)$ to obtain the particular solution to this nonhomogeneous ODE, giving the following:

$$\mathbf{x}_p(t) = \begin{pmatrix} \frac{1}{t^2} & t \\ -\frac{2}{t^3} & 1 \end{pmatrix} \begin{pmatrix} \frac{t^4}{2} - \frac{1}{2} - 3\ln(t) \\ 4t - 3 - \frac{1}{t^3} \end{pmatrix} = \begin{pmatrix} \frac{9t^4 - 6t^3 - 6\ln(t) - 3}{2t^2} \\ \frac{3t^4 - 3t^3 + 6\ln(t)}{t^3} \end{pmatrix}.$$

The homogeneous part satisfying the ICs gives:

$$\mathbf{x}_h(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 = \begin{pmatrix} \frac{1}{t^2} & t \\ -\frac{2}{t^3} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x10 \\ x20 \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{3t^2} + \frac{2t}{3}\right) x10 + \left(-\frac{1}{3t^2} + \frac{t}{3}\right) x20 \\ \left(-\frac{2}{3t^3} + \frac{2}{3}\right) x10 + \left(\frac{2}{3t^3} + \frac{1}{3}\right) x20 \end{pmatrix}.$$

It follows that the solution is given by:

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) = \begin{pmatrix} \left(\frac{1}{3t^2} + \frac{2t}{3}\right) x10 + \left(-\frac{1}{3t^2} + \frac{t}{3}\right) x20 + \frac{9t^4 - 6t^3 - 6\ln(t) - 3}{2t^2} \\ \left(-\frac{2}{3t^3} + \frac{2}{3}\right) x10 + \left(\frac{2}{3t^3} + \frac{1}{3}\right) x20 + \frac{3t^4 - 3t^3 + 6\ln(t)}{t^3} \end{pmatrix}.$$